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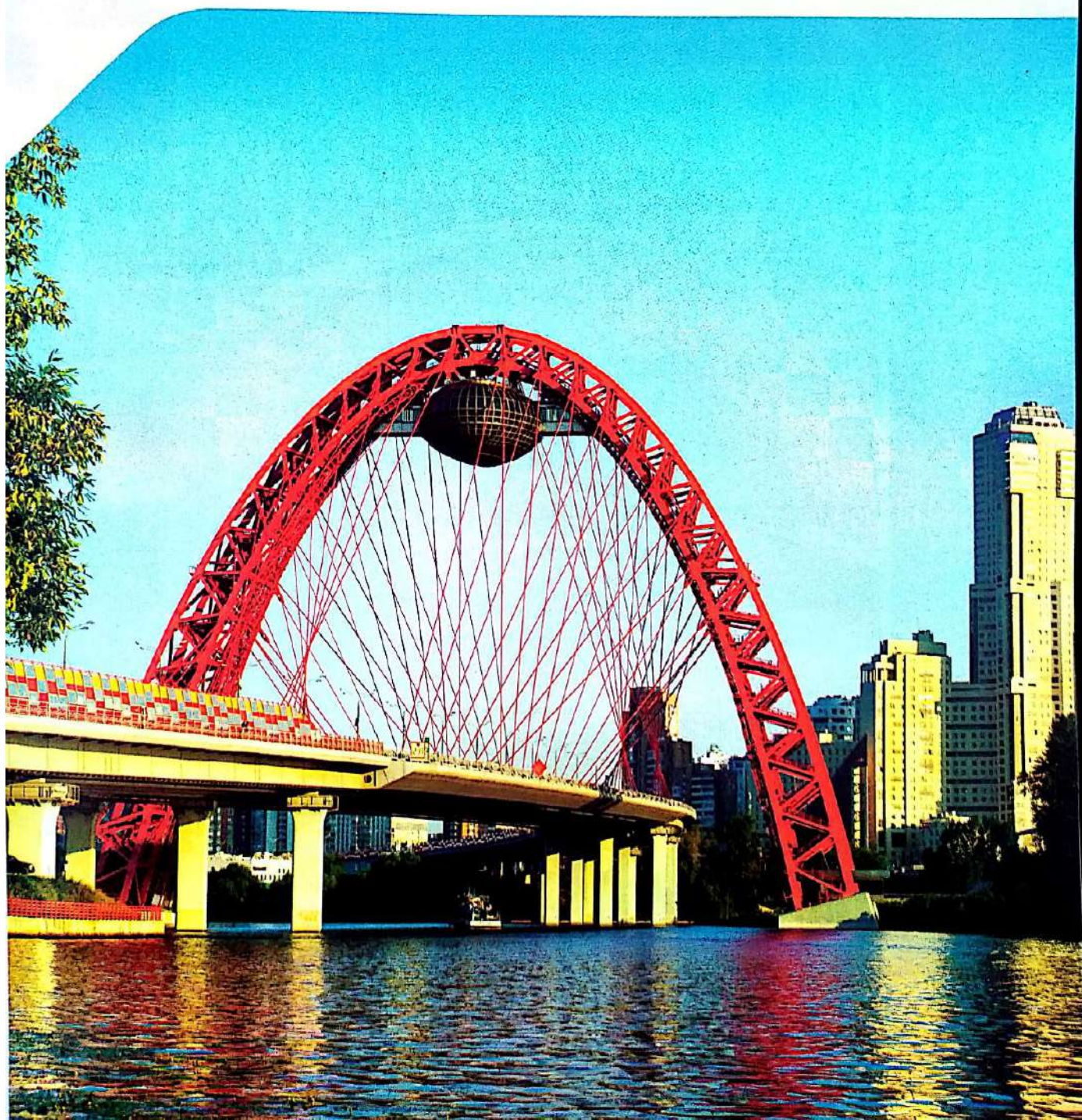
Distribution of maths syllabus

First year secondary – First term

Month	Algebra	Trigonometry	Geometry
	One period weekly	One period weekly	Two periods weekly
The rest of September and October	<ul style="list-style-type: none"> Solving quadratic equations in one variable. An introduction in complex numbers : Imaginary number - Integer powers of i - Equality of two complex numbers Operations on complex numbers (Adding, subtracting and multiplying complex numbers) - Conjugate numbers. 	<ul style="list-style-type: none"> Directed angle : Degree measure system - Standard position of directed angles - Positive and negative measures of a directed angle - angle's position in the orthogonal coordinate plane - Equivalent angles. Systems of measuring angle - Radian measure - Radian angle - Relation between degree measure and radian measure - The length of an arc of a circle. 	<ul style="list-style-type: none"> Similarity of polygons - Similar polygons - The scale factor of similarity of two polygons. Similarity of triangles - Postulate Corollary (1) - Corollary (2) Theorem (1) - Theorem (2)
November	<ul style="list-style-type: none"> Determining the types of roots of a quadratic equation - Discriminant. Relation between the two roots of the second degree equation and the coefficients of its terms - Sum and product of two roots. Forming the quadratic equation whose two roots are known - Forming the quadratic equation from the roots of another equation. 	<ul style="list-style-type: none"> Trigonometric functions - The unit circle - The basic trigonometric functions of an angle - Reciprocals of the basic trigonometric functions - Signs of trigonometric functions - Trigonometric ratios of some special angles. 	<ul style="list-style-type: none"> The ratio between the areas of two similar triangles - Theorem (3) The relation between the areas of two similar polygons (Theorem (4)) Applications of similarity in the circle : Well known problem - Corollary (1) - Converse of well known problem - Corollary (2) The triangle proportionality theorems : Parallel lines and proportional parts (theorem (1)) - Corollary - Converse of theorem (1) Talis' theorem (theorem (2)) - Talis' special theorem.
December	<ul style="list-style-type: none"> Sign of function : Sign of constant function - Sign of linear function - Sign of quadratic function. Quadratic inequalities in one variable - Solving quadratic inequality in one variable. 	<ul style="list-style-type: none"> Related angles. General solution of trigonometric equations in the form : $\sin \alpha = \cos \beta, \dots$ Graphing trigonometric functions : sine function and its properties - cosine function and its properties. Finding the measure of an angle given the value of one of its trigonometric ratios. 	<ul style="list-style-type: none"> Angle bisector and proportional parts : Bisector of an angle of a triangle (theorem (3)) - Finding the length of the interior and the exterior bisector of an angle of a triangle - Converse of theorem (3) - Fact. Applications of proportionality in the circle : Power of a point with respect to a circle - Secant, tangent and measures of angles - Investigating measure of the angle resulting from the intersection of a secant and a tangent (or two tangents) to a circle (well known problem).
January	Final revision + Holiday Christmas + Mid-year vacation		

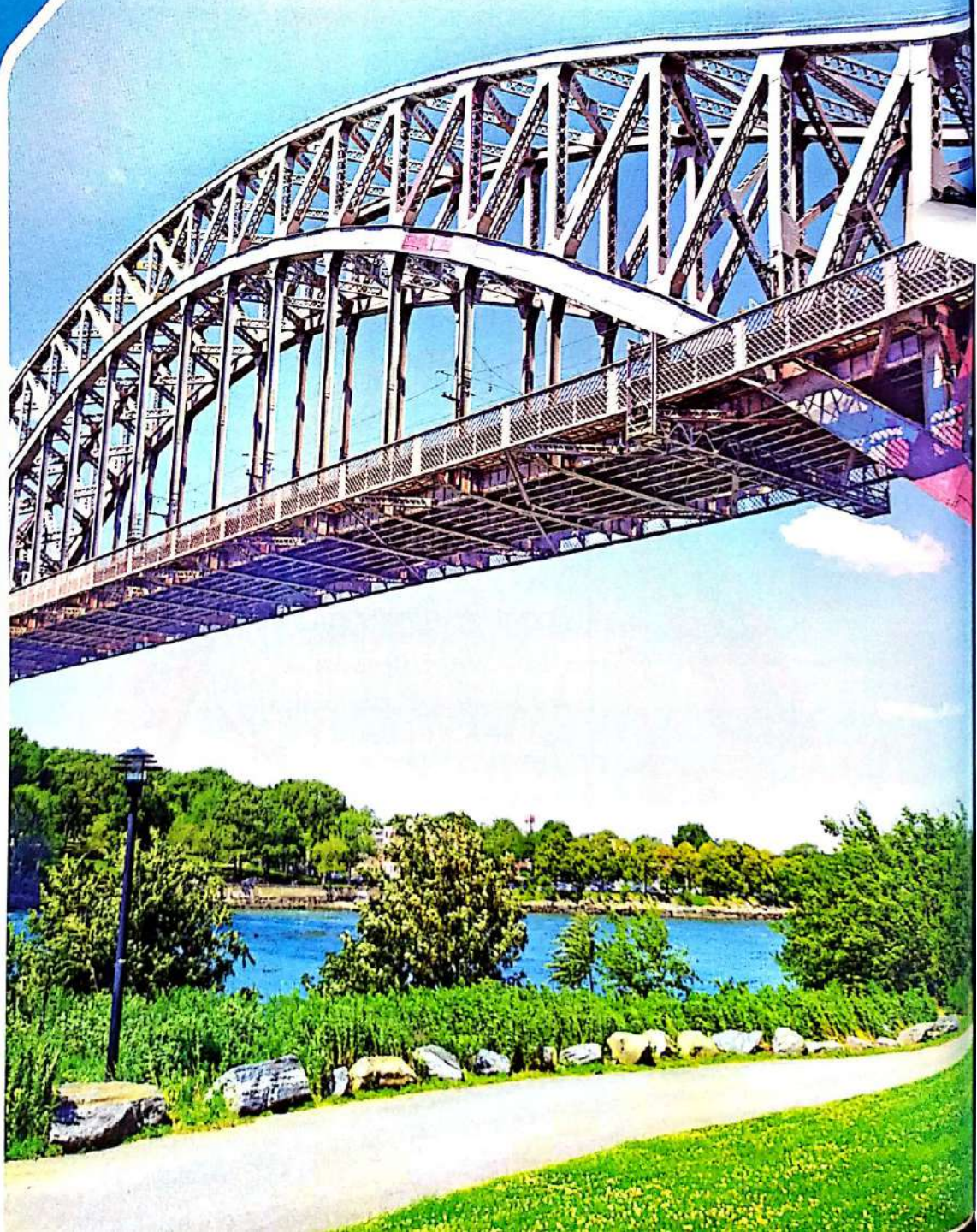
FIRST

Algebra and Trigonometry



UNIT

1



Algebra, Relations and Functions.

Unit Lessons

Lesson 1 : Solving quadratic equations in one variable.

Lesson 2 : An introduction in complex numbers.

Lesson 3 : Determining the types of roots of a quadratic equation.

Lesson 4 : Relation between the two roots of the second degree equation and the coefficients of its terms.

Lesson 5 : Forming the quadratic equation whose two roots are known.

Lesson 6 : Sign of a function.

Lesson 7 : Quadratic inequalities in one variable.

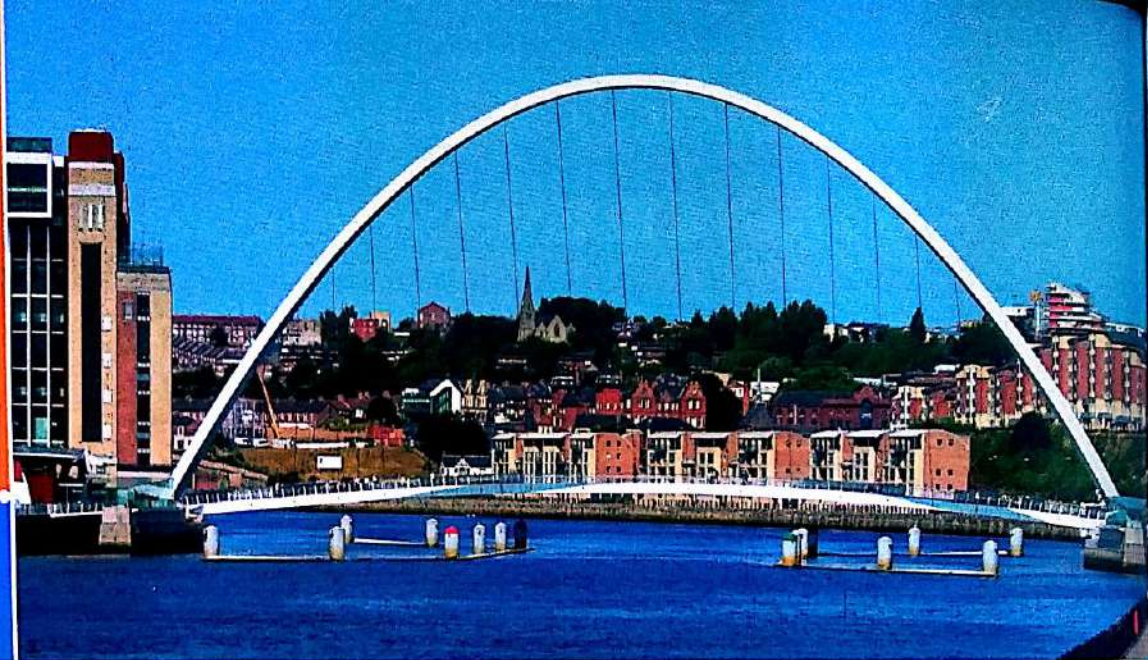
Unit Objectives

By the end of this unit, the student should be able to :

- Solve a quadratic equation in one variable algebraically and graphically.
- Use the quadratic equation in one variable to solve some life applications.
- Recognize an introduction in complex numbers (Definition of the complex number , integer powers of i and equality of two complex numbers).
- Carry out operations on the complex numbers.
- Recognize the two conjugate numbers in the complex numbers.
- Recognize the discriminant of the quadratic equation in one variable.
- Investigate the type of the two roots of the quadratic equation in one variable given the coefficients of its terms.
- Find the sum and the product of the two roots of a quadratic equation in one variable.
- Find some of the coefficients of terms of the quadratic equation in one variable in terms of one of the two roots or both of them.
- Form the quadratic equation in one variable whose roots are given.
- Form the quadratic equation in one variable given another quadratic equation in one variable.
- Investigate the sign of a function (constant - linear - quadratic).
- Solve quadratic inequalities in one variable.

Lesson

1



Solving quadratic equations in one variable

Preface

- Each of the equations : $X^2 + 4X - 12 = 0$, $3X^2 - 10X - 8 = 0$, $X^2 - 16 = 0$ is called an equation **of the second degree** in one variable X “because the greatest power of the variable X is 2” and the general form of the second degree equation in one variable is :

$aX^2 + bX + c = 0$, $a \neq 0$ It is called the **quadratic equation** in one variable.

- We have previously studied that the solution of the equation in \mathbb{R} means finding the values of the variable which satisfy the equation and belong to \mathbb{R} . Each of these solutions is called a **root of the equation**.
- The quadratic equation has **at most two solutions** in \mathbb{R}
- In this lesson we will revise together how to find the roots of the quadratic equation in \mathbb{R} algebraically and graphically.

First Solving the quadratic equation in one variable algebraically

We have previously studied that the quadratic equation in one variable could be solved algebraically by two methods :

By factorization.

By the general formula.

In the following , we will remember together both of the two methods.

1 By factorization

This method depends on factorizing the expression : $aX^2 + bX + c$ "if possible" , besides then we use the following property to find the two roots of the equation :

If a and b are two real numbers and if $a \times b = 0$, then either $a = 0$ or $b = 0$

2 By the general formula

This method depends on substituting in the general formula to find the roots of the equation $aX^2 + bX + c = 0$ where $a \neq 0$

This formula is $X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

, where a is the coefficient of X^2 , b is the coefficient of X and c is the absolute term.

The problems in the following example show the algebraic solution by the previous methods :

Example 1

Find in \mathbb{R} the solution set of each of the following equations :

1 $X^2 - 5X - 6 = 0$

2 $4X^2 = 25$

3 $X^2 + 6X + 9 = 0$

4 $X^2 - 2X - 6 = 0$

5 $X + \frac{5}{X} = 4$, $X \neq 0$

Solution

1 $\because X^2 - 5X - 6 = 0 \quad \therefore (X - 6)(X + 1) = 0$ "factorizing the trinomial"

\therefore Either $X - 6 = 0$ or $X + 1 = 0$

i.e. $X = 6$ or $X = -1$

\therefore The solution set = $\{6, -1\}$

2 $\because 4X^2 = 25 \quad \therefore 4X^2 - 25 = 0$

$\therefore (2X - 5)(2X + 5) = 0$ "factorizing the difference between two squares"

\therefore Either $2X - 5 = 0$ or $2X + 5 = 0$

i.e. $X = \frac{5}{2}$ or $X = -\frac{5}{2}$

\therefore The solution set = $\{\frac{5}{2}, -\frac{5}{2}\}$

3 $\because X^2 + 6X + 9 = 0$

$\therefore (X + 3)^2 = 0$ "factorizing the perfect square"

$\therefore X = -3$

Another solution

$\because 4X^2 = 25 \quad \therefore X^2 = \frac{25}{4} \quad \therefore X = \pm \sqrt{\frac{25}{4}}$

$\therefore X = \pm \frac{5}{2} \quad \therefore$ The solution set = $\{\frac{5}{2}, -\frac{5}{2}\}$

$\therefore X + 3 = 0$

\therefore The solution set = $\{-3\}$

Notice that

You can check your answer by substituting with the two solutions in the given equation.

4 The expression : $x^2 - 2x - 6$ is difficult to be factorized , so we use the general formula.

$$\therefore a = 1, b = -2, c = -6$$

$$\begin{aligned}\therefore x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times (-6)}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{4 + 24}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}\end{aligned}$$

$$\therefore \text{The solution set} = \{1 + \sqrt{7}, 1 - \sqrt{7}\}$$

5 $\therefore x + \frac{5}{x} = 4$ "By multiplying both sides of the equation by x "

$$\therefore x^2 + 5 = 4x$$

$$\therefore x^2 - 4x + 5 = 0 \quad \text{"Notice putting the equation in the form : } ax^2 + bx + c = 0\text{"}$$

$$\therefore a = 1, b = -4, c = 5$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$\therefore \sqrt{-4} \notin \mathbb{R} \quad \therefore \text{There is no real roots of the equation : } x^2 - 4x + 5 = 0$$

$$\therefore \text{The solution set} = \emptyset$$

TRY TO SOLVE

Find in \mathbb{R} the solution set of each of the following equations :

1 $x^2 - 5x + 6 = 0$

2 $5x^2 + 2x = 4$

3 $3x^2 = 27$

4 $x(x - 4) = 3$

Second Solving the quadratic equation in one variable graphically

To solve the quadratic equation in one variable graphically , we follow the following :

1 Put the equation on the form : $ax^2 + bx + c = 0$

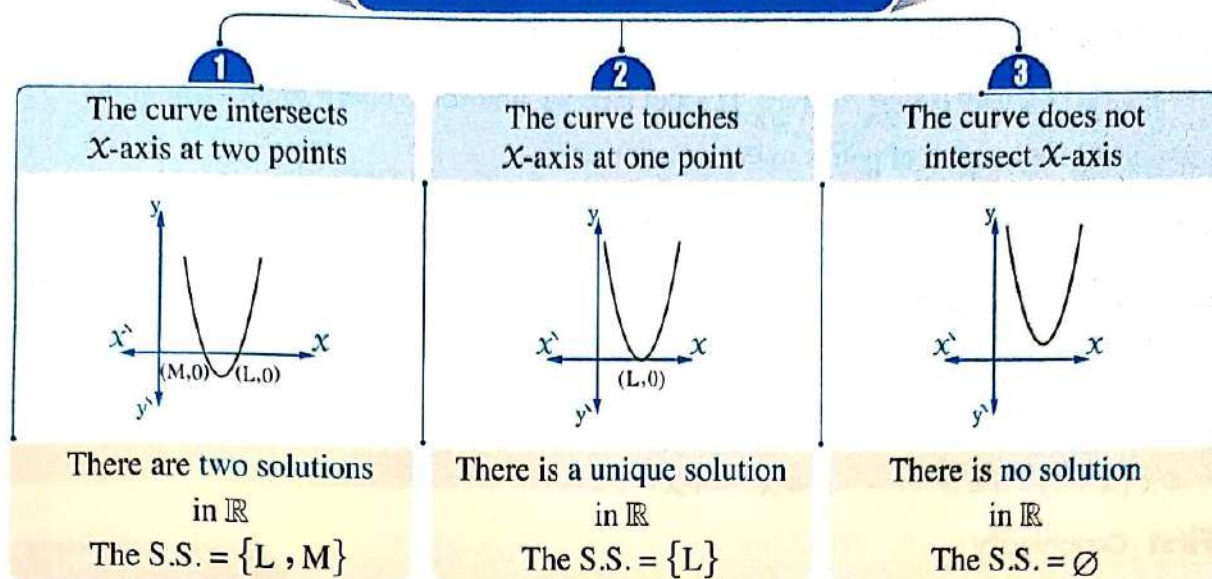
2 Let $f(x) = ax^2 + bx + c$

3 Graph the function f

4 Determine the points of intersection of the curve with the x -axis , then the x -coordinates of these intersection points are the solutions of the equation $f(x) = 0$

i.e. $ax^2 + bx + c = 0$

According to that, we have three cases



• The following examples show the previous cases :

Example 2

Find graphically in \mathbb{R} the S.S. of the equation :

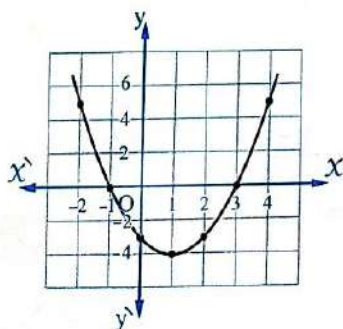
$$x^2 - 2x - 3 = 0$$

using the interval $[-2, 4]$

Solution

Let $f(x) = x^2 - 2x - 3$

x	-2	-1	0	1	2	3	4
y	5	0	-3	-4	-3	0	5



From the graph ,
the S.S. = $\{3, -1\}$

Example 3

Find graphically in \mathbb{R} the S.S. of the equation :

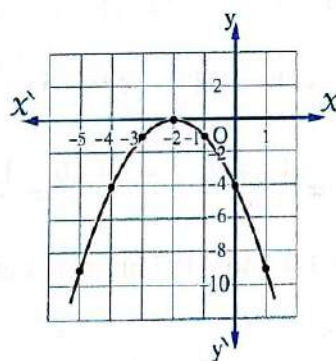
$$-x^2 - 4x - 4 = 0$$

using the interval $[-5, 1]$

Solution

Let $f(x) = -x^2 - 4x - 4$

x	-5	-4	-3	-2	-1	0	1
y	-9	-4	-1	0	-1	-4	-9



From the graph ,
the S.S. = $\{-2\}$

Example 4

Find graphically in \mathbb{R} the S.S. of the equation :

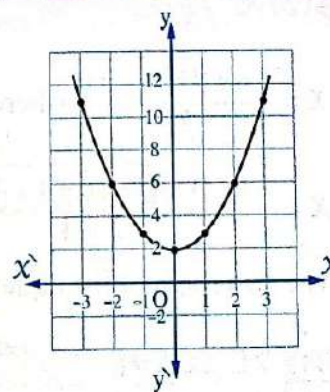
$$x^2 + 2 = 0$$

using the interval $[-3, 3]$

Solution

Let $f(x) = x^2 + 2$

x	-3	-2	-1	0	1	2	3
y	11	6	3	2	3	6	11



From the graph ,
the S.S. = \emptyset

Remark

In case of the interval is not given, then we can graph the function by finding the vertex of the curve which is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$, and then we find some points to the right of it, and the same number of points to the left of it.

Example 5

Solve graphically in \mathbb{R} the equation :

$4x(x-1)-5=0$, then verify the result algebraically "given that $\sqrt{6} \approx 2.4$ "

Solution

$$\therefore 4x(x-1)-5=0 \quad \therefore 4x^2-4x-5=0$$

First Graphically :

Let $f(x) = 4x^2 - 4x - 5$

• Find the vertex point of the curve :

$$\therefore \text{The } x\text{-coordinate of the vertex point} = \frac{-b}{2a} = \frac{4}{8} = \frac{1}{2}$$

$$, f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) - 5 = -6$$

$$\therefore \text{The vertex point of the curve is } \left(\frac{1}{2}, -6\right)$$

• Form the following table :

x	-1	0	$\left(\frac{1}{2}\right)$	1	2
y	3	-5	(-6)	-5	3

• From the graph we notice that :

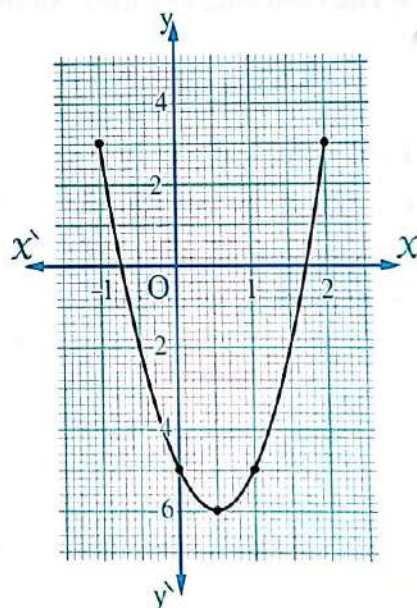
the roots are -0.7 and 1.7 approximately.

Second Algebraically :

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ where } a = 4, b = -4, c = -5$$

$$\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 4 \times (-5)}}{2 \times 4} = \frac{4 \pm \sqrt{96}}{8} = \frac{4 \pm 4\sqrt{6}}{8} = \frac{1 \pm \sqrt{6}}{2} \approx \frac{1 \pm 2.4}{2}$$

\therefore The two roots of the equation are 1.7 and -0.7 approximately.



TRY TO SOLVE

Solve graphically in \mathbb{R} the equation :

$x^2 - 4x + 4 = 0$, taking $x \in [0, 4]$, then verify the result algebraically.

Lesson

2



An introduction in complex numbers

Introduction

- We have studied before different sets of numbers , the set of natural numbers " \mathbb{N} " , the set of integers " \mathbb{Z} " , the set of rational numbers " \mathbb{Q} " , and the set of real numbers " \mathbb{R} "
- We have seen that every set comes as an extension of the preceeding one , to solve new kinds of equations which were not solved in the preceeding set.

For example :

- ▶ The equation : $x + 8 = 3$ has no solution in \mathbb{N} , thus we extend \mathbb{N} to \mathbb{Z}
- ▶ The equation : $2x = 1$ has no solution in \mathbb{Z} , thus we extend \mathbb{Z} to \mathbb{Q}
- ▶ The equation : $x^2 = 2$ has no solution in \mathbb{Q} , thus we extend \mathbb{Q} to \mathbb{R}

- There are , unfortunately , many problems that can not be solved by the use of real numbers alone. For example , we are unable to solve the equation $x^2 = -1$ There is no real number "a" such that $a^2 = -1$ Thus we must extend the set of real numbers \mathbb{R} to a new set of numbers to enable us to find the solution of the equation $x^2 = -1$

This new set is called THE SET OF COMPLEX NUMBERS , and before studying the set of complex numbers in details , we will firstly recognize the imaginary number "i".

The imaginary number "i"

The imaginary number "i" is defined as the number whose square is -1

i.e. $i^2 = -1$

Thus we can solve the equation : $x^2 = -1$ as follows :

$$\therefore x^2 = -1$$

$$\therefore x^2 = i^2$$

$$\therefore x = \pm \sqrt{i^2}$$

$$\therefore x = \pm i$$

\therefore The solution set = $\{i, -i\}$

Notice that

$$\bullet i \times i = i^2 = -1$$

$$\bullet -i \times -i = i^2 = -1$$

Remarks

- ▶ The number "i" does not belong to the set of real numbers.

i.e. $i \notin \mathbb{R}$, so it will not be represented by a point on the real number line.

- ▶ The numbers $3i, -2i, \sqrt{5}i, \dots$ are imaginary numbers.

- ▶ We can write the square roots of negative numbers as follows :

$$\sqrt{-2} = \sqrt{2i^2} = \sqrt{2}i, \sqrt{-3} = \sqrt{3i^2} = \sqrt{3}i, \sqrt{-25} = \sqrt{25i^2} = 5i \text{ and so on ...}$$

- ▶ The operations on the square roots can not be generalized on the imaginary numbers.

If a and b are two negative real numbers, then $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$

For example $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{-1 \times -1}$

because $\sqrt{-1} \times \sqrt{-1} = \sqrt{i^2} \times \sqrt{i^2} = i \times i = i^2 = -1$

but $\sqrt{-1 \times -1} = \sqrt{(-1)^2} = \sqrt{1} = 1$

Integer powers of "i"

The number "i" satisfies the rules of powers that you have studied in the preparatory stage and since $i^2 = -1$, then :

$$\bullet i^3 = i^2 \times i = -1 \times i = -i$$

$$\bullet i^4 = i^2 \times i^2 = -1 \times -1 = 1$$

$$\bullet i^5 = i^4 \times i = 1 \times i = i$$

$$\bullet i^6 = i^4 \times i^2 = 1 \times -1 = -1$$

$$\bullet i^7 = i^4 \times i^3 = 1 \times -i = -i$$

$$\bullet i^8 = i^4 \times i^4 = 1 \times 1 = 1 \dots \text{and so on.}$$

From this we find that :

- ▶ The integer powers of "i" give one of the values $i, -1, -i$ or 1

- ▶ These values are repeated if the power is increased by 4

Generally : For each $n \in \mathbb{Z}$,

$$\bullet i^{4n} = (i^4)^n = 1^n = 1$$

$$\bullet i^{4n+1} = i^{4n} \times i = 1 \times i = i$$

$$\bullet i^{4n+2} = i^{4n} \times i^2 = 1 \times -1 = -1$$

$$\bullet i^{4n+3} = i^{4n} \times i^3 = 1 \times -i = -i$$

$$\bullet i^{4n+4} = i^{4n} \times i^4 = 1 \times 1 = 1 \dots \text{and so on.}$$

* And the following table summarizes the previous :

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$
$i^5 = i$	$i^6 = -1$	$i^7 = -i$	$i^8 = 1$
$i^{4n+1} = i$	$i^{4n+2} = -1$	$i^{4n+3} = -i$	$i^{4n+4} = 1$

Generally

To find i^n where n is an integer

We find the remainder of the division $n \div 4$, if :

The remainder = 0 then $i^n = 1$

The remainder = 1 then $i^n = i$

The remainder = 2 then $i^n = i^2 = -1$

The remainder = 3 then $i^n = i^3 = -i$

And the following example shows the previous.

Example 1

Find each of the following in the simplest form :

1 i^{16}

2 i^{63}

3 i^{42}

4 i^{101}

5 i^{4n+23}

Solution

1 $\because 16 \div 4 = 4$, the remainder = 0

$\therefore i^{16} = 1$

2 $\because 63 \div 4 = 15$, the remainder = 3

$\therefore i^{63} = i^3 = -i$

3 $\because 42 \div 4 = 10$, the remainder = 2

$\therefore i^{42} = i^2 = -1$

4 $\because 101 \div 4 = 25$, the remainder = 1

$\therefore i^{101} = i$

5 $\because i^{4n+23} = i^{4n} \times i^{23}$

$\because i^{4n} = 1$, $i^{23} = i^3$ "because $23 \div 4 = 5$, the remainder = 3"

$\therefore i^{4n+23} = 1 \times i^3 = -i$

Remark

We can express 1 using the imaginary number i to integer powers from the multiples of 4 , and this helps in simplifying some of imaginary numbers , as in the following example.

Example 2

Find each of the following in the simplest form :

1 i^{-19}

2 $\frac{1}{i^5}$

Solution

1 $i^{-19} = i^{-19} \times i^{20}$
 $= i$

"Notice that : $i^{20} = 1$ "

2 $\frac{1}{i^5} = \frac{i^8}{i^5} = i^3 = -i$

TRY TO SOLVE

Find each of the following in the simplest form :

1 i^{40}

2 i^{35}

3 i^{4n+29}

4 i^{-23}

The complex number - The set of complex numbers

- When we solve the equation : $x^2 - 6x + 13 = 0$ using the general formula , we find that :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

$$\therefore \sqrt{-16} = \sqrt{16i^2} = 4i \quad \therefore x = \frac{6 \pm 4i}{2} \quad \therefore x = 3 \pm 2i$$

i.e. the equation : $x^2 - 6x + 13 = 0$ has no solution in \mathbb{R} , but it has two roots that do not belong to the real numbers which are $3 + 2i$ and $3 - 2i$

- Each of $3 + 2i$ and $3 - 2i$ is called "a complex number".

The complex number

The complex number is the number that can be written in the form $a + bi$

, where a and b are two real numbers and $i^2 = -1$

- a is called the real part.
- b is called the imaginary part.

Examples for complex numbers : $2 - i$, $7 + 13i$, $5i - 4$, $\sqrt{2} + \sqrt{3}i$

Remarks

If Z is a complex number where $Z = a + bi$, then :

1 If $b = 0$, then $Z = a$ and we say that Z is a real number.

Such as $Z = 5$ is a real number and it is a complex number.

2 If $a = 0$, then $Z = bi$ and we say that Z is an imaginary number. (where $b \neq 0$)

Such as $Z = 2i$ is an imaginary number and it is a complex number.

The set of complex numbers

The set of complex numbers \mathbb{C} is defined as $\mathbb{C} = \{a + bi : a \in \mathbb{R}, b \in \mathbb{R}, i^2 = -1\}$

Example 3

Find the solution set of each of the following equations in the set of complex numbers :

1 $x^2 + 16 = 0$

2 $2x^2 + 18 = 0$

3 $x^2 + x + 1 = 0$

Solution

1 $\therefore x^2 + 16 = 0$

$\therefore x^2 = -16$

$\therefore x = \pm\sqrt{-16}$

$\therefore x = \pm\sqrt{16i^2}$

$\therefore x = \pm 4i$

\therefore The solution set = $\{4i, -4i\}$

2 $\therefore 2x^2 + 18 = 0$

$\therefore 2x^2 = -18$

$\therefore x^2 = -9$

$\therefore x = \pm\sqrt{-9}$

$\therefore x = \pm\sqrt{9i^2}$

$\therefore x = \pm 3i$

\therefore The solution set = $\{3i, -3i\}$

3 $\therefore a = 1, b = 1, c = 1$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$$

\therefore The solution set = $\left\{\frac{-1}{2} + \frac{\sqrt{3}}{2}i, \frac{-1}{2} - \frac{\sqrt{3}}{2}i\right\}$

TRY TO SOLVE

Find the solution set of each of the following equations in the set of complex numbers :

1 $5x^2 + 180 = 0$

2 $x^2 - 2x + 5 = 0$

Equality of two complex numbers

Two complex numbers are equal if and only if the two real parts are equal and the two imaginary parts are equal.

i.e. If $(a + bi)$ and $(c + di)$ are two complex numbers and if $a = c$, $b = d$, then $a + bi = c + di$

and vice versa If $a + bi = c + di$, then $a = c$, $b = d$

Example 4

Find the values of x and y which satisfy the equation : $x - 3y + (2x + y)i = 6 + 5i$

Solution

\therefore The two complex numbers are equal.

$$\therefore x - 3y = 6 \quad (1)$$

$$2x + y = 5 \quad (2)$$

Multiply the equation (2) by 3 :

$$\therefore 6x + 3y = 15 \quad (3)$$

$$\text{By adding (1) and (3) :} \quad \therefore 7x = 21$$

$$\therefore x = 3$$

$$\text{By substituting in (2) :} \quad \therefore y = -1$$

TRY TO SOLVE

Find the values of x and y which satisfy the equation :

$$4x - y + (2x + y)i = 5 + 7i$$

Operations on the complex numbers

- When we add, subtract and multiply complex numbers, we use the commutative, associative and distribution properties as we do in algebraic expressions.
- When adding or subtracting two complex numbers, we add or subtract real parts together and add or subtract imaginary parts together.

The following examples show how to carry out some operations on the complex numbers.

Example 5

Find the result of each of the following in the simplest form :

1 $(3 + 7i) + (5 - 9i)$

2 $(2 - 4i) - (5 - i)$

3 $(4 + 3i)(2 - 5i)$

4 $(5 - 2i)(5 + 2i)$

5 $(3 + 2i)^2$

6 $(1 - i)^4$

Solution

1 $(3 + 7i) + (5 - 9i) = (3 + 5) + (7i - 9i)$ (Commutative and associative properties)

$= 8 - 2i$

2 $(2 - 4i) - (5 - i) = (2 - 4i) + (-5 + i) = (2 - 5) + (-4i + i) = -3 - 3i$

3 $(4 + 3i)(2 - 5i) = 4(2 - 5i) + 3i(2 - 5i)$ (Distribution property)

$= 8 - 20i + 6i - 15i^2$

$= 8 - 20i + 6i + 15$ (where $i^2 = -1$)

$= (8 + 15) + (-20i + 6i) = 23 - 14i$

Notice that You can solve directly by using direct multiplication.

4 $(5 - 2i)(5 + 2i) = 25 - 4i^2$

$= 25 + 4$ (where $i^2 = -1$)

$= 29$

5 $(3 + 2i)^2 = 9 + 12i + 4i^2$

$= 9 + 12i - 4$ (where $i^2 = -1$)

$= 5 + 12i$

6 $(1 - i)^4 = ((1 - i)^2)^2 = (1 - 2i + i^2)^2 = (1 - 2i - 1)^2$

$= (-2i)^2 = 4i^2 = -4$

Remember that

$(a + b)(a - b) = a^2 - b^2$

Remember that

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

TRY TO SOLVE

Find the result of each of the following in the simplest form :

1 $(2 + 5i) + (-3 - 4i)$

2 $(2 - i)(2 + i)$

3 $(2 + 3i)(5 - i)$

4 $i(5 - 3i)$

The two conjugate numbers

The two numbers $a + bi$ and $a - bi$ are called conjugate numbers.

Note : Take care that the complex number and its conjugate differ only in the sign of their imaginary parts.

For example : The two numbers $3 + 4i$, $3 - 4i$ are conjugate numbers.

Remarks

- ▶ The conjugate of the number $2i - 5$ is the number $-2i - 5$ not $2i + 5$
- ▶ The conjugate of the number $2i$ is $-2i$
- ▶ The conjugate of the number 3 is 3
- ▶ The sum of the two conjugate numbers is always a real number , and the product of the two conjugate numbers is always a real number.

For example The complex number $3 + 4i$ its conjugate is $3 - 4i$, then :

* Their sum $= (3 + 4i) + (3 - 4i) = (3 + 3) + (4i - 4i) = 6 \in \mathbb{R}$

* Their product $= (3 + 4i)(3 - 4i) = 9 - 16i^2 = 9 + 16 = 25 \in \mathbb{R}$

TRY TO SOLVE

Write the conjugate of $5 - 4i$, then find :

- 1 The sum of the number and its conjugate.
- 2 The product of the number and its conjugate.

Example 6

Simplify to the simplest form :

1 $\frac{10}{3+i}$

2 $\frac{3+2i}{2-5i}$

3 $\frac{(2+i)(1-i)}{(1+i)(3-2i)}$

Solution

Notice : To simplify the fraction whose denominator is a complex number , we multiply its two terms by the conjugate of denominator.

1 \because The conjugate of the denominator is $(3 - i)$

$$\therefore \frac{10}{3+i} = \frac{10(3-i)}{(3+i)(3-i)} = \frac{10(3-i)}{9-i^2} = \frac{10(3-i)}{9+1} = \frac{10(3-i)}{10} = 3-i$$

$$2 \quad \frac{3+2i}{2-5i} = \frac{(3+2i)(2+5i)}{(2-5i)(2+5i)} = \frac{6+15i+4i+10i^2}{4-25i^2} \text{ but } i^2 = -1$$

$$\therefore \frac{3+2i}{2-5i} = \frac{6+19i-10}{4+25} = \frac{-4+19i}{29} = \frac{-4}{29} + \frac{19}{29}i$$

$$3 \quad \frac{(2+i)(1-i)}{(1+i)(3-2i)} = \frac{2-2i+i-i^2}{3-2i+3i-2i^2} = \frac{2-i+1}{3+i+2} = \frac{3-i}{5+i}$$

$$\therefore \frac{3-i}{5+i} = \frac{(3-i)(5-i)}{(5+i)(5-i)} = \frac{15-8i-1}{25-i^2} = \frac{14-8i}{26} = \frac{2(7-4i)}{26}$$

$$\therefore \frac{(2+i)(1-i)}{(1+i)(3-2i)} = \frac{7}{13} - \frac{4}{13}i$$

TRY TO SOLVE

Find each of the following in its simplest form :

$$1 \quad \frac{2+i}{3-4i}$$

$$2 \quad \frac{(2+i)(3+i)}{(2-i)(3-i)}$$

Example 7

If $x = \frac{7-i}{2-i}$ and $y = \frac{13-i}{4+i}$

Prove that : x and y are conjugate numbers , then prove that : $x^2 + y^2 = 16$

Solution

$$\therefore x = \frac{7-i}{2-i} = \frac{(7-i)(2+i)}{(2-i)(2+i)} = \frac{14+7i-2i-i^2}{4-i^2} = \frac{14+5i+1}{4+1} = \frac{15+5i}{5} = 3+i$$

$$\therefore y = \frac{13-i}{4+i} = \frac{(13-i)(4-i)}{(4+i)(4-i)} = \frac{52-13i-4i-i^2}{16-i^2} = \frac{52-17i-1}{16+1} = \frac{51-17i}{17} = 3-i$$

$\therefore x$ and y are conjugate numbers " **Notice that** the signs of the imaginary parts are different."

$$\therefore x^2 = (3+i)^2 = 9+6i+i^2 = 8+6i$$

$$\therefore y^2 = (3-i)^2 = 9-6i+i^2 = 8-6i$$

$$\therefore x^2 + y^2 = (8+6i) + (8-6i) = (8+8) + (6i-6i) = 16$$

TRY TO SOLVE

Prove that a and b are conjugate numbers if : $a = \frac{1-2i}{1-3i}$ and $b = \frac{2-i}{3-i}$

Example 8

Find the values of x and y that satisfy the equation : $\frac{(3+2i)(2-i)}{3+i} = x + yi$

Solution

$$\text{L.H.S.} = \frac{(3+2i)(2-i)}{3+i} = \frac{6+i-2i^2}{3+i} = \frac{6+i+2}{3+i} = \frac{8+i}{3+i}$$

Multiply the numerator and the denominator by the conjugate of the denominator ,
which is $(3-i)$

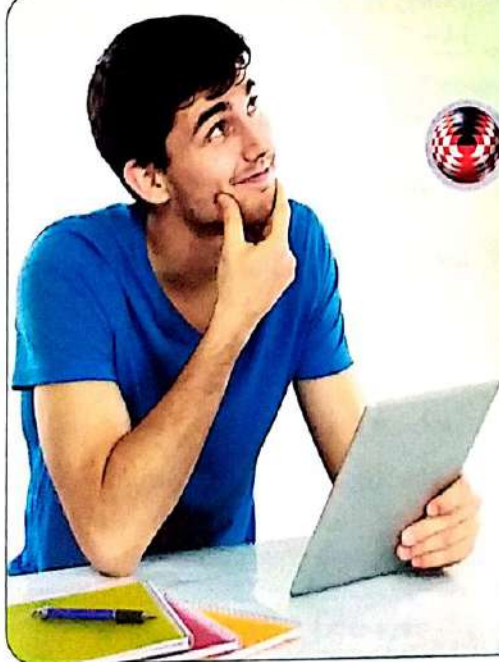
$$\therefore \text{L.H.S.} = \frac{8+i}{3+i} = \frac{(8+i)(3-i)}{(3+i)(3-i)} = \frac{24-5i-i^2}{9-i^2} = \frac{24-5i+1}{9+1} = \frac{25-5i}{10} = \frac{5}{2} - \frac{1}{2}i$$

$$\therefore x = \frac{5}{2} \text{ and } y = -\frac{1}{2}$$

TRY TO SOLVE

If $\frac{(3+i)(3-i)}{2+i} = x + yi$, find the values of : x and y

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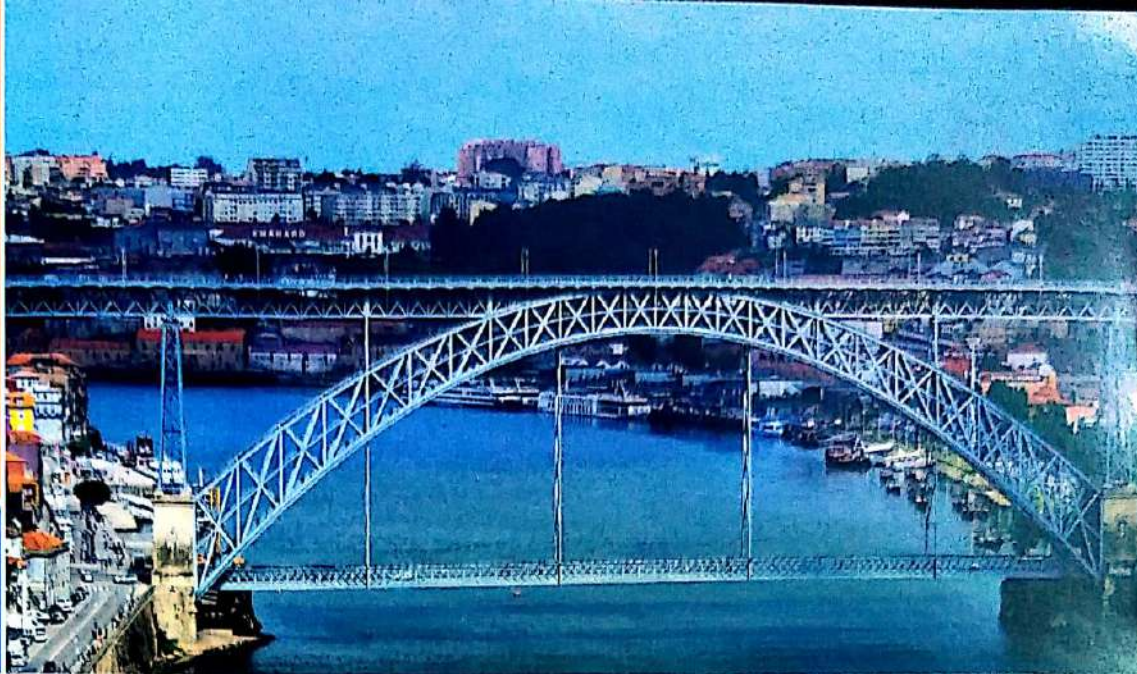


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Lesson

3



Determining the types of roots of a quadratic equation

- You have previously studied how to solve the second degree equation (the quadratic equation) in one variable in \mathbb{R} , and you have known that when solving it, we have two roots at most.
- In this lesson, we will determine the types of the two roots of the quadratic equation without solving it.

Discriminant

- Using the formula in solving the quadratic equation : $aX^2 + bX + c = 0$, where $a \neq 0$, we get two roots : $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$
- Both of these two roots include the expression : $\sqrt{b^2 - 4ac}$
- The expression : $b^2 - 4ac$ is called the discriminant of the quadratic equation because it is used to determine the types of roots of the quadratic equation as follows :

Discriminant	positive $(b^2 - 4ac) > 0$	equal to zero $b^2 - 4ac = 0$	negative $(b^2 - 4ac) < 0$
Type of the two roots	Two different real roots	Two equal real roots	Two complex and non real roots
A sketch for the function related to the equation			

Example 1

Determine the type of the two roots of each of the following equations :

1 $x^2 - 3x + 5 = 0$

2 $x^2 + 10x + 25 = 0$

3 $3x^2 + 10x = 4$

Solution

1 $\therefore a = 1, b = -3, c = 5$

\therefore The discriminant $= b^2 - 4ac = (-3)^2 - 4 \times 1 \times 5 = -11$ (negative quantity)

\therefore The two roots are complex and non real.

2 $\therefore a = 1, b = 10, c = 25$

\therefore The discriminant $= b^2 - 4ac = (10)^2 - 4 \times 1 \times 25 = 0$

\therefore The two roots are real and equal.

3 $\therefore 3x^2 + 10x - 4 = 0$

$\therefore a = 3, b = 10, c = -4$

\therefore The discriminant $= b^2 - 4ac = (10)^2 - 4 \times 3 \times (-4) = 148$ (positive quantity)

\therefore The two roots are different and real.

TRY TO SOLVE

Determine the type of the two roots of each of the following equations :

1 $x^2 - 7x + 10 = 0$

2 $x^2 + 4x + 5 = 0$

3 $4x^2 - 12x = -9$

Example 2

Prove that the two roots of the equation : $7x^2 - 11x + 5 = 0$ are two complex and non real roots, then use the formula to find these two roots.

Solution

$\therefore a = 7, b = -11, c = 5$

\therefore The discriminant $= b^2 - 4ac = (-11)^2 - 4 \times 7 \times 5 = -19 < 0$

\therefore The two roots are complex and non real roots.

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{11 \pm \sqrt{-19}}{14} = \frac{11 \pm \sqrt{19}i}{14}$$

\therefore The two roots of the equation are $\frac{11 + \sqrt{19}i}{14}, \frac{11 - \sqrt{19}i}{14}$

TRY TO SOLVE

If $x^2 - 4x + 5 = 0$, then prove that the two roots are complex and not real, then use the general formula to find these two roots.

Example 3

If the two roots of the equation : $x^2 - kx + 2k - 4x + 5 = 0$ are equal, then find the real values of k and find these two roots.

Solution

Put the equation on the general form

$$\therefore x^2 - (k + 4)x + (2k + 5) = 0$$

$$\therefore \text{The discriminant} = (k + 4)^2 - 4 \times 1 \times (2k + 5) = k^2 + 8k + 16 - 8k - 20 = k^2 - 4$$

\therefore The two roots of the equation are equal

$$\therefore \text{The discriminant} = 0$$

$$\therefore k^2 - 4 = 0$$

$$\therefore k^2 = 4$$

$$\therefore k = \pm 2$$

$$\therefore \text{at } k = 2$$

$$\therefore \text{The equation is } x^2 - 6x + 9 = 0$$

$$\therefore (x - 3)^2 = 0$$

$$\therefore x = 3$$

at $k = 2$ the two roots are equal, each one = 3

$$\therefore \text{at } k = -2$$

$$\therefore \text{The equation is } x^2 - 2x + 1 = 0$$

$$\therefore (x - 1)^2 = 0$$

$$\therefore x = 1$$

at $k = -2$ the two roots are equal, each one = 1

TRY TO SOLVE

Find the real value of k which makes the two roots of the equation :

$4x^2 - 8x + k = 0$ equal and find these two roots.

Example 4

1 Find the real values of m which satisfy that the equation : $x^2 - (2m - 1)x + m^2 = 0$ has no real roots (i.e., has no solutions in \mathbb{R})

2 Find the real values of k which satisfy that the equation : $x^2 + 2(k - 1)x + k^2 = 0$ has two real roots (i.e., has solutions in \mathbb{R})

Solution

1 \therefore The equation does not have real roots $\therefore b^2 - 4ac < 0$

$$\therefore (2m - 1)^2 - 4m^2 < 0$$

$$\therefore 4m^2 - 4m + 1 - 4m^2 < 0$$

$$\therefore -4m < -1$$

$$\therefore m > \frac{1}{4}$$

\therefore The equation has no real roots if $m \in \left] \frac{1}{4}, \infty \right[$

2 \therefore The equation has two real roots

\therefore The two roots are either different or equal

$$\therefore b^2 - 4ac \geq 0$$

$$\therefore 4(k-1)^2 - 4 \times 1 \times k^2 \geq 0$$

$$\therefore 4k^2 - 8k + 4 - 4k^2 \geq 0$$

$$\therefore -8k \geq -4$$

$$\therefore k \leq \frac{1}{2}$$

\therefore The equation has two real roots if $k \in \left] -\infty, \frac{1}{2} \right]$

TRY TO SOLVE

If the equation : $m^2 x^2 + (2m - 2)x + 1 = 0$ has no roots in \mathbb{R} , find the real values of m

Example 5

Prove that for all real values of a , there is no real roots for the equation :

$$4x^2 - 12ax + 9a^2 + 4 = 0$$

Solution

The discriminant = $(-12a)^2 - 4(4)(9a^2 + 4)$

$$= 144a^2 - 144a^2 - 64 = -64 \text{ (is negative quantity for all values of } a)$$

\therefore There is no real roots of the equation.

Remark

If the coefficients a , b and c in the quadratic equation : $ax^2 + bx + c = 0$ are rational numbers and the discriminant is a perfect square, then the roots are real rational numbers.

For example :

The equation : $3x^2 - 5x - 2 = 0$

• The coefficients : $a = 3$, $b = -5$, $c = -2$ are rational numbers

• The discriminant = $b^2 - 4ac = (-5)^2 - 4(3)(-2) = 25 + 24 = 49$ (is a perfect square)

\therefore The two roots are real rational numbers.

• **To verify that :** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-2)}}{2 \times 3} = \frac{5 \pm \sqrt{49}}{6} = \frac{5 \pm 7}{6}$

$$\therefore x = \frac{5+7}{6} = 2 \text{ or } x = \frac{5-7}{6} = -\frac{1}{3}$$

\therefore The two roots of the equation are $2, -\frac{1}{3}$ (are real rational numbers)

As for the equation : $x^2 - 2\sqrt{5}x + 1 = 0$

- (Coefficient of x) = $-2\sqrt{5}$ (isn't a rational number)
- And although the discriminant = $b^2 - 4ac = (-2\sqrt{5})^2 - 4(1)(1) = 20 - 4 = 16$ is a perfect square, however we find that its roots are real and not rational numbers.

• To verify that :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2\sqrt{5}) \pm \sqrt{(-2\sqrt{5})^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{2\sqrt{5} \pm \sqrt{16}}{2} = \frac{2\sqrt{5} \pm 4}{2}$$

$$\therefore x = \frac{2\sqrt{5} + 4}{2} = \sqrt{5} + 2 \text{ or } x = \frac{2\sqrt{5} - 4}{2} = \sqrt{5} - 2$$

\therefore The two roots $\sqrt{5} + 2$, $\sqrt{5} - 2$ are real and not rational numbers.

Example 6

If a and b are rational numbers,

prove that the two roots of the equation : $ax^2 + (a^2 + b^2)x + ab^2 = 0$ are rational.

Solution

$$\begin{aligned} \therefore \text{The discriminant} &= (a^2 + b^2)^2 - 4 \times a \times ab^2 = a^4 + 2a^2b^2 + b^4 - 4a^2b^2 \\ &= a^4 - 2a^2b^2 + b^4 = (a^2 - b^2)^2 \text{ is a perfect square} \end{aligned}$$

\therefore The coefficients are rational numbers and the discriminant is a perfect square

\therefore The two roots of the equation are rational.

TRY TO SOLVE

If a is a rational number, prove that the two roots of the equation :

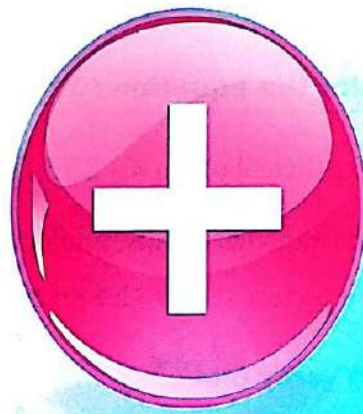
$$15x^2 - (10 + 3a)x + 2a = 0 \text{ are rational.}$$

Remark

If the discriminant of the quadratic equation isn't positive, then the two roots of the quadratic equation are complex numbers and conjugate.

Lesson

4



Relation between the two roots of the second degree equation and the coefficients of its terms

We know that the two roots of the quadratic equation : $aX^2 + bX + c = 0$ are :

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \text{ then :}$$

$$1 \text{ The sum of the two roots} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}$$

i.e. The sum of the two roots = $\frac{-\text{Coefficient of } X}{\text{Coefficient of } X^2}$

$$2 \text{ The product of the two roots} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

i.e. The product of the two roots = $\frac{\text{Absolute term}}{\text{Coefficient of } X^2}$

In a symbolic form , we write :

If L and M are the two roots of the quadratic equation : $aX^2 + bX + c = 0$, then :

$$1 \quad L + M = \frac{-b}{a}$$

$$2 \quad LM = \frac{c}{a}$$

Example 1

Without solving the equation, find the sum and the product of the two roots of each of the following equations:

1 $2x^2 + 5x - 12 = 0$

2 $6x^2 - 11x = 10$

Solution

1 $\therefore a = 2, \quad b = 5, \quad c = -12$

$$\therefore \text{The sum of the two roots} = \frac{-b}{a} = \frac{-5}{2}$$

$$\text{, the product of the two roots} = \frac{c}{a} = \frac{-12}{2} = -6$$

Check the solution with noticing that the two roots are

$$\frac{3}{2} \text{ and } -4$$

2 $\therefore 6x^2 - 11x = 10$

$$\therefore 6x^2 - 11x - 10 = 0$$

$$\therefore a = 6, \quad b = -11, \quad c = -10 \quad \therefore \text{The sum of the two roots} = \frac{-b}{a} = \frac{-(-11)}{6} = \frac{11}{6}$$

$$\text{, the product of the two roots} = \frac{c}{a} = \frac{-10}{6} = \frac{-5}{3}$$

TRY TO SOLVE

Complete: If $3x^2 + 5 = 4x$, then:

The sum of the two roots = and the product of the two roots =

Example 2

1 If the sum of the two roots of the equation: $2x^2 + kx + 1 = 0$ is $\frac{-3}{2}$, then find the value of k , and solve the equation in the set of complex numbers.

2 If the product of the two roots of the equation: $2x^2 - 4x + k = 0$ is $4\frac{1}{2}$, then find the value of k , and solve the equation in the set of complex numbers.

Solution

1 $\therefore \text{The sum of the two roots} = \frac{-3}{2}$

$$\therefore \frac{-k}{2} = \frac{-3}{2}$$

$$\therefore k = 3$$

$$\therefore \text{The equation is } 2x^2 + 3x + 1 = 0$$

$$\therefore (2x + 1)(x + 1) = 0$$

$$\therefore x = -\frac{1}{2} \quad \text{or} \quad x = -1$$

2 \therefore The product of the two roots $= 4 \frac{1}{2} = \frac{9}{2} \therefore \frac{k}{2} = \frac{9}{2} \therefore k = 9$

\therefore The equation is $2x^2 - 4x + 9 = 0 \therefore a = 2, b = -4, c = 9$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 2 \times 9}}{2 \times 2} = \frac{4 \pm \sqrt{-56}}{4} = \frac{4 \pm \sqrt{56}i}{4} = \frac{4 \pm 2\sqrt{14}i}{4} = 1 \pm \frac{\sqrt{14}}{2}i$$

$\therefore x = 1 + \frac{\sqrt{14}}{2}i$ or $x = 1 - \frac{\sqrt{14}}{2}i$

TRY TO SOLVE

- 1 If the sum of the two roots of the equation : $2x^2 - ax + 6 = 0$ is $3\frac{1}{2}$, then find the value of a , and solve the equation in the set of complex numbers.
- 2 If the product of the two roots of the equation : $x^2 + 3x + a = 0$ is 5, then find the value of a , and solve the equation in the set of complex numbers.

Example 3

- 1 If $x = -3$ is one of the two roots of the equation : $2x^2 + kx - 3 = 0$, then find the other root, and find the value of k
- 2 If $x = 6$ is one of the two roots of the equation : $x^2 - 5x + k = 0$, then find the other root, and find the value of k
- 3 If -1 and 5 are the two roots of the equation : $ax^2 + bx - 5 = 0$, then find the value of each of a and b

Solution

1 \therefore The product of the two roots $= \frac{c}{a} = \frac{-3}{2} \therefore -3 \times \text{the other root} = \frac{-3}{2}$

\therefore The other root $= \frac{-3}{2} \times \frac{1}{-3} \therefore$ The other root $= \frac{1}{2}$

\therefore The sum of the two roots $= \frac{-b}{a} = \frac{-k}{2},$

\therefore The two roots are $-3, \frac{1}{2} \therefore -3 + \frac{1}{2} = \frac{-k}{2}$

$\therefore \frac{-5}{2} = \frac{-k}{2} \therefore k = 5$

Another solution :

$\therefore X = -3$ is one of the roots of the equation : $2X^2 + kX - 3 = 0$

\therefore It satisfies it.

$$\therefore 2(-3)^2 + k(-3) - 3 = 0$$

$$\therefore 18 - 3k - 3 = 0$$

$$\therefore k = 5$$

\therefore The equation is : $2X^2 + 5X - 3 = 0$

$$\therefore (2X - 1)(X + 3) = 0$$

$$\therefore 2X - 1 = 0, \text{ then } X = \frac{1}{2}$$

$$\text{or } X + 3 = 0, \text{ then } X = -3$$

$$\therefore \text{The other root} = \frac{1}{2}$$

$$2 \therefore \text{The sum of the two roots} = \frac{-b}{a} = \frac{-(-5)}{1} = 5$$

$$\therefore 6 + \text{the other root} = 5$$

$$\therefore \text{The other root} = -1$$

$$\therefore \text{The product of the two roots} = \frac{c}{a} = \frac{k}{1} = k,$$

$$\therefore \text{The two roots are } 6, -1$$

$$\therefore 6 \times (-1) = k$$

$$\therefore k = -6$$

* Try to solve this example by another method as in 1

$$3 \therefore \text{The product of the two roots} = \frac{c}{a}$$

$$\therefore -1 \times 5 = \frac{-5}{a}$$

$$\therefore -5 = \frac{-5}{a}$$

$$\therefore a = 1$$

$$\therefore \text{The sum of the two roots} = \frac{-b}{a}$$

$$\therefore -1 + 5 = \frac{-b}{1}$$

$$\therefore 4 = -b$$

$$\therefore b = -4$$

Another solution :

$\therefore -1$ is a root of the equation.

$$\therefore a(-1)^2 + b(-1) - 5 = 0$$

$$\therefore a - b - 5 = 0$$

$$(1)$$

$\therefore 5$ is a root of the equation.

$$\therefore a(5)^2 + b(5) - 5 = 0$$

$$\therefore 25a + 5b - 5 = 0 \text{ "Divide by 5"}$$

$$\therefore 5a + b - 1 = 0$$

$$(2)$$

Adding the equations (1) and (2) :

$$\therefore 6a - 6 = 0$$

$$\therefore a = 1$$

By substituting in (1) :

$$\therefore 1 - b - 5 = 0$$

$$\therefore b = -4$$

TRY TO SOLVE

Find the other root of each of the following equations, then find the value of k in each case :

1 If $X = -1$ is one of the two roots of the equation : $X^2 + kX - 7 = 0$

2 If $X = \frac{5}{3}$ is one of the two roots of the equation : $9X^2 - 9X + k = 0$

Example 4

If $(1 + \sqrt{2}i)$ is one of the two roots of the equation : $X^2 - 2X + c = 0$ where $c \in \mathbb{R}$, then find :

1 The other root.

2 The value of c

Solution

$$\therefore \text{The sum of the two roots} = \frac{-(-2)}{1} = 2$$

$$\therefore (1 + \sqrt{2}i) + \text{the other root} = 2$$

$$\therefore \text{The other root} = 2 - (1 + \sqrt{2}i)$$

i.e. The other root $= 1 - \sqrt{2}i$

$$\therefore \text{The product of the two roots} = c$$

$$\therefore 1^2 - (\sqrt{2}i)^2 = c$$

$$\therefore 1 + 2 = c$$

$$\therefore (1 - \sqrt{2}i)(1 + \sqrt{2}i) = c$$

$$\therefore 1 - 2i^2 = c$$

$$\therefore c = 3$$

Notice that

The other root is the conjugate of the given root.

i.e. it equals $(1 - \sqrt{2}i)$

Another solution :

$$\therefore (1 + \sqrt{2}i) \text{ is one of the two roots of the given equation.}$$

$$\therefore \text{It satisfies the equation.}$$

$$\therefore 1 + 2\sqrt{2}i + (\sqrt{2}i)^2 - 2 - 2\sqrt{2}i + c = 0$$

$$\therefore -3 + c = 0$$

$$\therefore (1 + \sqrt{2}i)^2 - 2(1 + \sqrt{2}i) + c = 0$$

$$\therefore 1 + 2\sqrt{2}i - 2 - 2 - 2\sqrt{2}i + c = 0$$

$$\therefore c = 3$$

i.e. $X^2 - 2X + 3 = 0$

We can use the general formula to find the required other root.

TRY TO SOLVE

If $(\sqrt{2} + i)$ is one of the two roots of the equation : $x^2 - 2\sqrt{2}x + c = 0$ where $c \in \mathbb{R}$, then find :

1 The other root.

2 The value of c

Remarks

In the quadratic equation : $ax^2 + bx + c = 0$

1 If $a = 1$, then $L + M = -b$ and $LM = c$

i.e. The sum of the two roots = the additive inverse of the coefficient of x ,
the product of the two roots = the absolute term.

2 If $b = 0$, then $L + M = 0$, **i.e.** $L = -M$

i.e. One of the two roots of the equation is the additive inverse of the other.

3 If $a = c$, then $LM = 1$, **i.e.** $L = \frac{1}{M}$

i.e. One of the two roots of the equation is the multiplicative inverse of the other.

Example 5

1 Find the value of k , if one of the roots of the equation : $3x^2 + (k-3)x + 7 = 0$ is the additive inverse of the other root.

2 Find the value of k , if one of the roots of the equation : $2kx^2 + 7x + k^2 + 1 = 0$ is the multiplicative inverse of the other.

Solution

1 \therefore One of the roots is the additive inverse of the other

$$\therefore b = 0$$

$$\therefore k - 3 = 0$$

$$\therefore k = 3$$

2 \therefore One of the roots is the multiplicative inverse of the other

$$\therefore a = c$$

$$\therefore k^2 + 1 = 2k$$

$$\therefore k^2 - 2k + 1 = 0$$

$$\therefore (k-1)^2 = 0$$

$$\therefore k = 1$$

TRY TO SOLVE

Complete :

1 If one of the two roots of the equation : $x^2 + (k - 5)x - 9 = 0$

is the additive inverse of the other , then $k = \dots\dots\dots$

2 If one of the two roots of the equation : $x^2 + 3x + c = 0$

is the multiplicative inverse of the other , then $c = \dots\dots\dots$

Example 6

Find the value of d , if one of the two roots of the equation : $x^2 + d x - 50 = 0$ is double the additive inverse of the other root.

Solution

Let one of the two roots = L

\therefore The other root = $-2L$

\therefore The product of the two roots = $\frac{\text{absolute term}}{\text{coefficient of } x^2}$

$\therefore L(-2L) = \frac{-50}{1}$

$\therefore -2L^2 = -50$

$\therefore L = \pm 5$

\therefore The sum of the two roots = $\frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$

$\therefore L + (-2L) = \frac{-d}{1}$

$\therefore -L = -d$

$\therefore L = d$

$\therefore d = \pm 5$

Example 7

Find the value of n , if one of the two roots of the equation : $x^2 + 15 = n x$ is less than the twice of the other root by 1

Solution

The equation is $x^2 - n x + 15 = 0$

Let one of the two roots be L

\therefore The other root = $2L - 1$

\therefore The product of the two roots = 15

$\therefore L(2L - 1) = 15$

$\therefore 2L^2 - L - 15 = 0$

$\therefore (2L + 5)(L - 3) = 0$

$\therefore L = \frac{-5}{2} \text{ or } L = 3$

$$\text{At } L = \frac{-5}{2}$$

$$\therefore \text{The other root} = 2 \times \frac{-5}{2} - 1 = -6$$

$$\therefore n = \text{the sum of the two roots} = \frac{-5}{2} - 6 = \frac{-17}{2}$$

$$\text{At } L = 3$$

$$\therefore \text{The other root} = 2 \times 3 - 1 = 5$$

$$\therefore n = 3 + 5 = 8$$

TRY TO SOLVE

Find the value of k , if one of the two roots of the equation : $x^2 - kx + 12 = 0$ is three times the other root.

Example 8

Find the satisfying condition which makes one of the two roots of the equation : $ax^2 + bx + c = 0$ equal to the additive inverse of twice the other root.

Solution

Let one of the two roots be L

$$\therefore \text{The other root} = -2L$$

$$\therefore \text{The sum of the two roots} = \frac{-b}{a}$$

$$\therefore L + (-2L) = \frac{-b}{a}$$

$$\therefore L = \frac{b}{a} \quad (1)$$

$$\therefore \text{The product of the two roots} = \frac{c}{a}$$

$$\therefore L \times (-2L) = \frac{c}{a}$$

$$\therefore L^2 = \frac{-c}{2a} \quad (2)$$

By substituting from (1) in (2) :

$$\therefore \left(\frac{b}{a}\right)^2 = \frac{-c}{2a}$$

$$\therefore \frac{b^2}{a^2} = \frac{-c}{2a}$$

$$\therefore \frac{b^2}{a} = \frac{-c}{2}$$

$$\therefore 2b^2 + ac = 0 \quad (\text{That is the required condition})$$

TRY TO SOLVE

Find the satisfying condition which makes one of the two roots of the equation : $ax^2 + bx + c = 0$ equal to four times the other root.

Lesson

5



Forming the quadratic equation whose two roots are known

Let L and M be the two roots of the quadratic equation : $a x^2 + b x + c = 0$

By multiplying the two sides by $\frac{1}{a}$ where $a \neq 0$, the equation becomes in the form :

$$x^2 + \frac{b}{a} x + \frac{c}{a} = 0 \quad \text{i.e.} \quad x^2 - \left(\frac{-b}{a}\right) x + \frac{c}{a} = 0 \quad (1)$$

$$\text{But } \frac{-b}{a} = L + M, \quad \frac{c}{a} = LM$$

By substituting in (1), we get the quadratic equation whose roots are L, M which is :

$$x^2 - (L + M)x + LM = 0 \quad (2)$$

$$\text{i.e.} \quad x^2 - (\text{the sum of the two roots}) x + \text{the product of the two roots} = 0$$

And by factorizing the trinomial in the left side of the equation (2), we get another form of the last equation which is $(x - L)(x - M) = 0$

Example 1

Form the quadratic equation whose roots are :

1 $\frac{3}{2}, \frac{5}{4}$

2 $3 + \sqrt{2}, 3 - \sqrt{2}$

3 $\frac{-1+i}{i}, \frac{2}{1+i}$

Solution

1 The sum of the two roots $= \frac{3}{2} + \frac{5}{4} = \frac{11}{4}$, the product of them $= \frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$
 \therefore The equation is $x^2 - (\text{the sum of the two roots}) x + \text{the product of the two roots} = 0$

\therefore The equation is $x^2 - \frac{11}{4} x + \frac{15}{8} = 0$ (by multiplying by 8)

\therefore The equation is $8x^2 - 22x + 15 = 0$

- 2 The sum of the two roots $= 3 + \sqrt{2} + 3 - \sqrt{2} = 6$
 , The product of the two roots $= (3 + \sqrt{2})(3 - \sqrt{2}) = 7$
 \therefore The equation is $x^2 - 6x + 7 = 0$

- 3 $\therefore \frac{-1+i}{i} = \frac{(-1+i)i}{i \times i} = \frac{-i+i^2}{i^2} = \frac{-i-1}{-1} = 1+i$
 $\therefore \frac{2}{1+i} = \frac{2(1-i)}{(1+i)(1-i)} = \frac{2-2i}{1-i^2} = \frac{2-2i}{2} = 1-i$
 \therefore The sum of the two roots $= 1+i+1-i = 2$
 , The product of the two roots $= (1+i)(1-i) = 2$
 \therefore The equation is $x^2 - 2x + 2 = 0$

TRY TO SOLVE

Form the quadratic equation whose roots are :

1 $-4, 7$

2 $3-2i, \frac{4+7i}{2+i}$

Forming a quadratic equation from the roots of another equation**Example 2**

If the two roots of the equation : $x^2 - 5x - 6 = 0$ are L, M , find the equation whose roots are $L+7, M+7$

Solution

The required in this example is forming an equation using a given equation where there is a certain relation between the roots of the two equations. There are many methods for solving this example and we will mention them in the following :

The first method

- 1 Find the two roots of the given equation.
- 2 Find the two roots of the required equation.
- 3 Form the required equation.

$$\therefore x^2 - 5x - 6 = 0$$

$$\therefore (x-6)(x+1) = 0$$

$\therefore 6, -1$ are the two roots of the given equation.

Let $L = 6$, $M = -1$, the two roots of the required equation be D , E

$$\therefore D = L + 7 = 6 + 7 = 13, E = M + 7 = -1 + 7 = 6$$

$$\therefore D + E = 13 + 6 = 19, DE = 13 \times 6 = 78$$

$$\therefore \text{The required equation is } x^2 - 19x + 78 = 0$$

The second method

Let D and E be the two roots of the required equation

$$\therefore D = L + 7, E = M + 7$$

$$\therefore D + E = L + 7 + M + 7 = L + M + 14$$

$$\therefore L + M = 5 \text{ (from the given equation)}$$

$$\therefore D + E = 5 + 14 = 19$$

$$DE = (L + 7)(M + 7) = LM + 7(L + M) + 49$$

$$\therefore LM = -6 \text{ (from the given equation)}$$

$$\therefore DE = -6 + 7 \times 5 + 49 = 78$$

$$\therefore \text{The required equation is } x^2 - 19x + 78 = 0$$

The third method

Let D and E be the two roots of the required equation

$$\therefore D = L + 7, E = M + 7$$

$$\therefore L = D - 7, M = E - 7$$

$$\therefore L \text{ is one of the two roots of the given equation : } x^2 - 5x - 6 = 0$$

$$\therefore L^2 - 5L - 6 = 0$$

$$\therefore L = D - 7$$

$$\therefore (D - 7)^2 - 5(D - 7) - 6 = 0$$

$$\therefore D^2 - 14D + 49 - 5D + 35 - 6 = 0$$

$$\therefore D^2 - 19D + 78 = 0$$

i.e. D is a root of the equation : $x^2 - 19x + 78 = 0$ (which is the required equation)

Remark

The third method is used only if the relation between the first root of the given equation and the first root of the required equation is the same relation between the second root of the given equation and the second root of the required equation.

Remember the following identities

$$1 \quad L^2 + M^2 = (L + M)^2 - 2LM$$

$$2 \quad (L - M)^2 = (L + M)^2 - 4LM$$

$$3 \quad L^3 + M^3 = (L + M) [(L + M)^2 - 3LM]$$

$$4 \quad L^3 - M^3 = (L - M) [(L + M)^2 - LM]$$

$$5 \quad \frac{1}{M} + \frac{1}{L} = \frac{L + M}{LM}$$

$$6 \quad \frac{L}{M} + \frac{M}{L} = \frac{L^2 + M^2}{LM} = \frac{(L + M)^2 - 2LM}{LM}$$

Example 3

If L, M are the two roots of the equation : $x^2 - 7x + 9 = 0$ where $L > M$, find the numerical value of each of the following expressions :

1 $L^2 + M^2$

2 $L^2 + 3LM + M^2$

3 $L - M$

4 $L^3 - M^3$

Solution

$\therefore L, M$ are the two roots of the equation : $x^2 - 7x + 9 = 0 \quad \therefore L + M = 7$ and $LM = 9$

1 $L^2 + M^2 = (L + M)^2 - 2LM = (7)^2 - 2 \times 9 = 49 - 18 = 31$

2 $L^2 + 3LM + M^2 = (L^2 + 2LM + M^2) + LM = (L + M)^2 + LM = (7)^2 + 9 = 49 + 9 = 58$

3 $(L - M)^2 = (L + M)^2 - 4LM = (7)^2 - 4 \times 9 = 49 - 36 = 13$

$$\therefore L - M = \sqrt{13}, \text{ where } L > M$$

4 $L^3 - M^3 = (L - M) [(L + M)^2 - LM]$

by substituting from (3) :

$$\therefore L^3 - M^3 = \sqrt{13} (7^2 - 9) = \sqrt{13} (49 - 9) = 40\sqrt{13}$$

Example 4

If the two roots of the equation : $x^2 - 8x + 5 = 0$ are L and M

, form the equation whose roots are $\frac{1}{L}$ and $\frac{1}{M}$

Solution

$\therefore L$ and M are the two roots of the given equation. $\therefore L + M = 8$ and $LM = 5$

$\therefore \frac{1}{L}$ and $\frac{1}{M}$ are the two roots of the required equation.

$$\therefore \text{The sum of the two roots} = \frac{1}{L} + \frac{1}{M} = \frac{M+L}{LM} = \frac{8}{5}$$

$$\text{, the product of the two roots} = \frac{1}{L} \times \frac{1}{M} = \frac{1}{LM} = \frac{1}{5}$$

$$\therefore \text{The required equation is } x^2 - \frac{8}{5}x + \frac{1}{5} = 0$$

i.e. $5x^2 - 8x + 1 = 0$

Example 5

If L and M are the two roots of the equation :

$x^2 - 5x + 9 = 0$, find the equation whose roots are L^2 and M^2

Solution

$\therefore L$ and M are the two roots of the given equation. $\therefore L + M = 5$ and $LM = 9$

$\therefore L^2$ and M^2 are the two roots of the required equation.

\therefore The sum of the two roots $= L^2 + M^2 = (L + M)^2 - 2LM = 5^2 - 2 \times 9 = 7$

, the product of the two roots $= L^2 \times M^2 = (LM)^2 = 9^2 = 81$

\therefore The required equation is $x^2 - 7x + 81 = 0$

Example 6

If L and M are the two roots of the equation :

$3x^2 + 5x - 7 = 0$, find the equation whose roots are $L + \frac{1}{M}$, $M + \frac{1}{L}$

Solution

$\therefore L$ and M are the two roots of the given equation.

$\therefore L + M = -\frac{5}{3}$ and $LM = -\frac{7}{3}$

, $\therefore L + \frac{1}{M}$, $M + \frac{1}{L}$ are the two roots of the required equation.

\therefore The sum of the two roots $= L + \frac{1}{M} + M + \frac{1}{L} = L + M + \frac{L + M}{LM}$

$$= -\frac{5}{3} + \frac{\frac{-5}{3}}{-\frac{7}{3}} = -\frac{5}{3} + \frac{5}{7} = \frac{-35 + 15}{21} = -\frac{20}{21}$$

, the product of the two roots $= \left(L + \frac{1}{M}\right) \left(M + \frac{1}{L}\right) = LM + \frac{1}{LM} + 2$

$$= -\frac{7}{3} - \frac{3}{7} + 2 = \frac{-49 - 9 + 42}{21} = -\frac{16}{21}$$

\therefore The required equation is $x^2 - \frac{-20}{21}x + \frac{-16}{21} = 0$

i.e. $21x^2 + 20x - 16 = 0$

TRY TO SOLVE

If L, M are the two roots of the equation :

$$2x^2 - 3x - 1 = 0, \text{ find the equation whose roots are } L^2, M^2$$

Example 7

If $\frac{2}{L}, \frac{2}{M}$ are the two roots of the equation : $x^2 - 6x + 4 = 0$,

find the equation whose roots are L, M

Solution

$\therefore \frac{2}{L}, \frac{2}{M}$ are the two roots of the given equation.

$$\therefore \frac{2}{L} \times \frac{2}{M} = 4$$

$$\therefore \frac{4}{LM} = 4$$

$$\therefore LM = 1$$

$$\therefore \frac{2}{L} + \frac{2}{M} = 6$$

$$\therefore \frac{2L + 2M}{LM} = 6$$

$$\therefore \frac{2(L + M)}{1} = 6$$

$$\therefore L + M = \frac{6}{2} = 3$$

$\therefore L$ and M are the two roots of the required equation, $L + M = 3, LM = 1$

\therefore The required equation is $x^2 - 3x + 1 = 0$

TRY TO SOLVE

If $\frac{1}{L}$ and $\frac{1}{M}$ are the two roots of the equation : $6x^2 - 5x + 1 = 0$,

find the equation whose roots are L and M

Example 8

If the difference between the two roots of the equation : $x^2 - kx + 4k = 0$ equals three times the product of the two roots of the equation : $x^2 - 3x - k = 0$, find the value of k

Solution

Let L and M be the two roots of the equation : $x^2 - kx + 4k = 0$

$$\therefore L + M = k, \quad LM = 4k$$

\therefore The difference between L and M equals three times the product of the two roots of the equation : $x^2 - 3x - k = 0$

$$\therefore L - M = -3k$$

$$\therefore (L - M)^2 = (L + M)^2 - 4LM \text{ (from the previous identities)}$$

$$\therefore (-3k)^2 = k^2 - 4(4k) \quad \therefore 9k^2 = k^2 - 16k \quad \therefore 8k^2 + 16k = 0$$

$$\therefore 8k(k + 2) = 0 \quad \therefore k = 0 \text{ or } k + 2 = 0 \quad \therefore k = -2$$

Another solution :

By using the law of the difference between the two roots :

$$\therefore L - M = \frac{\pm \sqrt{\text{the discriminant}}}{a} = \frac{\pm \sqrt{b^2 - 4ac}}{a} \text{ and from the equation :}$$

$$x^2 - kx + 4k = 0, \text{ we found that :}$$

$$L - M = \pm \sqrt{k^2 - 16k} \quad (1)$$

, $\therefore L - M$ equals three times the product

$$\text{of the two roots of : } x^2 - 3x - k = 0$$

$$\therefore L - M = -3k \quad (2)$$

, from (1), (2) :

$$\therefore \pm \sqrt{k^2 - 16k} = -3k, \text{ by squaring both sides}$$

$$\therefore k^2 - 16k = 9k^2 \quad \therefore 8k^2 + 16k = 0 \quad \therefore k = 0 \text{ or } k = -2$$

Remark

It is possible to deduce the law of the difference between the two roots from the general formula with the same method used for finding the sum of the two roots in the previous lesson.

TRY TO SOLVE

If the difference between the two roots of the equation : $x^2 + kx + 2k = 0$ equals twice the product of the two roots of the equation : $6x^2 + 5x - k = 0$, find the value of k

Lesson

6



Sign of a function

Remember that

You have studied in the preparatory stage that :

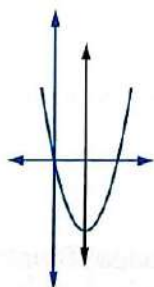
The function from set X to set Y is a relation which relates each element of X to only one element of Y and :

- The set X is called the domain of the function.
 - The set Y is called the codomain of the function.
 - The set of images of the elements of the domain X is called the range of the function and this range is a subset of the codomain Y
- Also , we have studied that each function is a relation but not any relation is a function , and we can determine whether the relation is a function or not , by using (the vertical line test)

The vertical line test

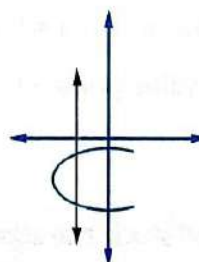
If we draw a vertical line intersecting the curve representing the relation

at one point



then the relation is a function

at two points or more



then the relation is not a function

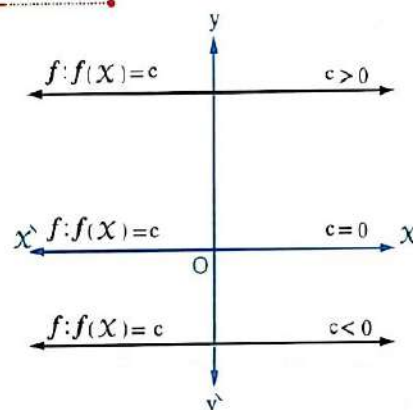
- You have also studied the graphical representation for each of the constant function ,
the first degree function (linear function)
and the second degree function (quadratic function)

Now we are going to remind you with some facts for each function :

The constant function

$f : f(X) = c$ ($c \in \mathbb{R}$) is represented by a straight line parallel to the X -axis and passes through the point $(0, c)$ and this line :

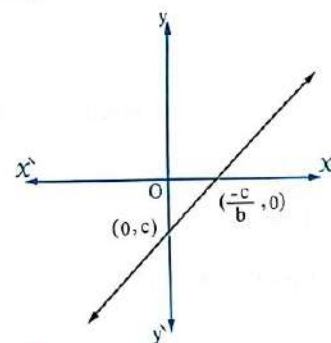
- * Is above the X -axis if $c > 0$
- * Coincides to the X -axis if $c = 0$
- * Is under the X -axis if $c < 0$



The linear function

$f : f(X) = bX + c$ (where $b \neq 0$) is represented by a straight line and this line :

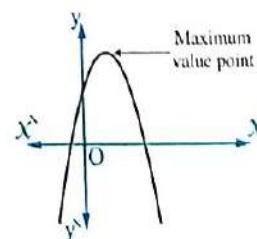
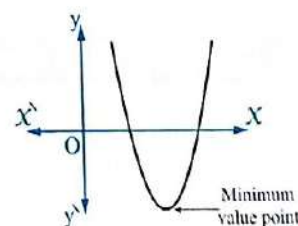
- * Intersects the y -axis at the point $(0, c)$
- * Intersects the X -axis at the point $(-\frac{c}{b}, 0)$



The quadratic function

$f : f(X) = aX^2 + bX + c$ (where $a \neq 0$) is represented by a curve symmetrical about a straight line parallel to the y -axis , and this curve :

- * Is open upwards if $a > 0$ and the function has a minimum value point.
- * Is open downwards if $a < 0$ and the function has a maximum value point.



In this lesson , we will study the sign of each of the three previous functions.

Investigating the sign of a function

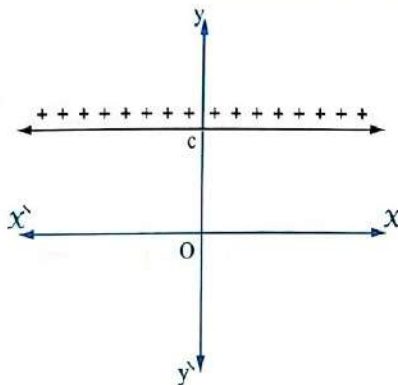
Investigating the sign of a function is to determine the values of x at which the values of the function f are as follows :

- Positive , **i.e.** $f(x) > 0$
- Negative , **i.e.** $f(x) < 0$
- Equal to zero , **i.e.** $f(x) = 0$

First The sign of the constant function

The following figures represent the two functions :

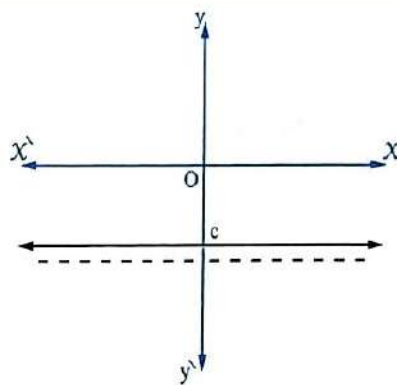
$$f : f(x) = c \text{ (where } c \text{ is positive)}$$



We notice that

The function is positive for all $x \in \mathbb{R}$

$$f : f(x) = c \text{ (where } c \text{ is negative)}$$



We notice that

The function is negative for all $x \in \mathbb{R}$

From the previous, we deduce that :

The sign of the constant function $f : f(x) = c$, $c \in \mathbb{R}^*$ is the same sign of $c \forall x \in \mathbb{R}$

Notice that

The symbol \forall means "for every"

For example :

- If $f(x) = 5$, then the sign of the function f is positive $\forall x \in \mathbb{R}$
- If $f(x) = -3$, then the sign of the function f is negative $\forall x \in \mathbb{R}$

TRY TO SOLVE

Determine the sign of each of the following two functions :

1 $f : f(x) = 10$

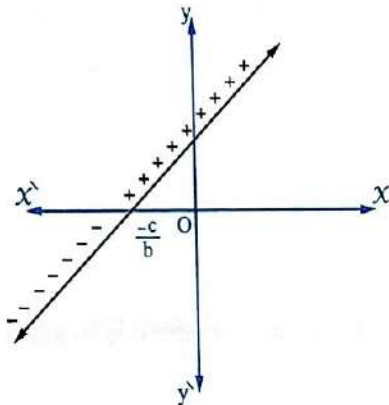
2 $f : f(x) = -\frac{2}{5}$

Second

The sign of the first degree function (linear function)

The following figures represent the two functions :

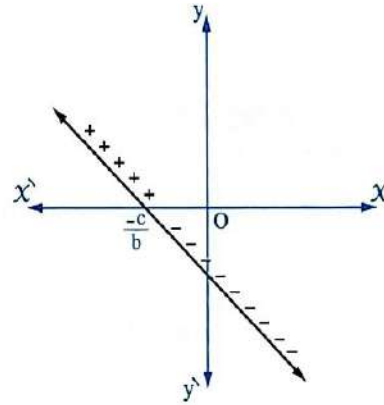
$f : f(X) = bX + c$ (b is positive)



We notice that the sign of the function :

- is the same as the sign of b (**positive**)
at $X > \frac{-c}{b}$
- is opposite to the sign of b (**negative**)
at $X < \frac{-c}{b}$
- equals **zero** at $X = \frac{-c}{b}$

$f : f(X) = bX + c$ (b is negative)



We notice that the sign of the function :

- is the same as the sign of b (**negative**)
at $X > \frac{-c}{b}$
- is opposite to the sign of b (**positive**)
at $X < \frac{-c}{b}$
- equals **zero** at $X = \frac{-c}{b}$

From the previous , we deduce that :

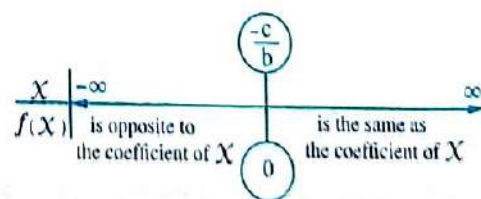
To find the sign of the linear function $f : f(X) = bX + c$, $b \neq 0$, we put $f(X) = 0$

$\therefore bX + c = 0 \qquad \therefore X = \frac{-c}{b}$

\therefore The sign of the function f :

- 1 Is the same as the sign of b at $X > \frac{-c}{b}$
- 2 Is opposite to the sign of b at $X < \frac{-c}{b}$
- 3 $f(X) = 0$ at $X = \frac{-c}{b}$

And we illustrate this on the opposite number line.



Example 1

Determine the sign of each of the following two functions using the number line :

1 $f : f(x) = 3x + 6$

2 $f : f(x) = 1 - \frac{1}{2}x$

Solution

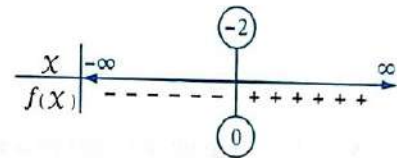
1 $\because f(x) = 3x + 6$ put $f(x) = 0$

$\therefore 3x + 6 = 0$ $\therefore x = -2$

\therefore The sign of the function f is :

- positive at $x > -2$
- negative at $x < -2$
- $f(x) = 0$ at $x = -2$

We illustrate the solution on the opposite number line.



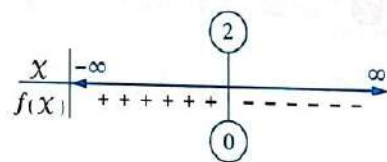
2 $\because f(x) = -\frac{1}{2}x + 1$ put $f(x) = 0$

$\therefore -\frac{1}{2}x = -1$ $\therefore x = 2$

\therefore The sign of the function f is :

- negative at $x > 2$
- positive at $x < 2$
- $f(x) = 0$ at $x = 2$

We illustrate the solution on the opposite number line.

**TRY TO SOLVE**

Determine the sign of each of the following two functions :

1 $f : f(x) = -3x + 6$

2 $f : f(x) = 2 + \frac{1}{2}x$

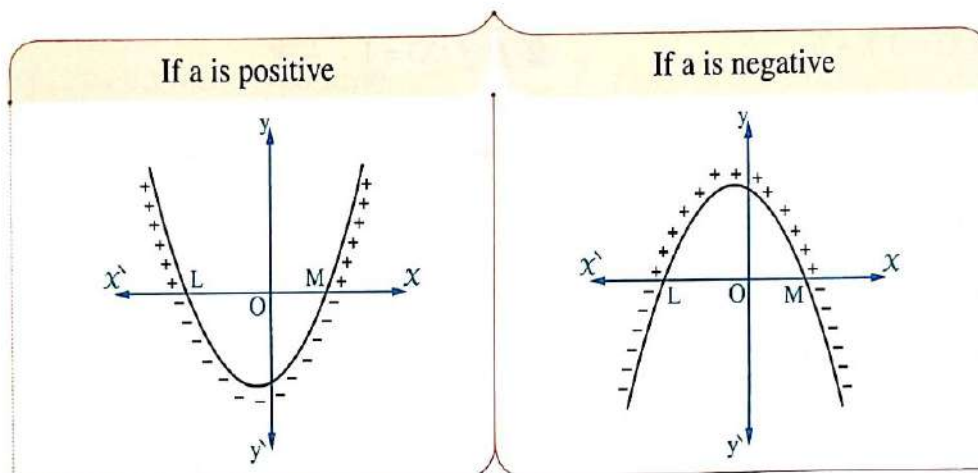
Third The sign of the second degree function (quadratic function)

To determine the sign of the quadratic function $f : f(x) = ax^2 + bx + c$, $a \neq 0$

, we have to obtain the discriminant of the equation : $ax^2 + bx + c = 0$, there are three cases :

1 The discriminant: $b^2 - 4ac > 0$

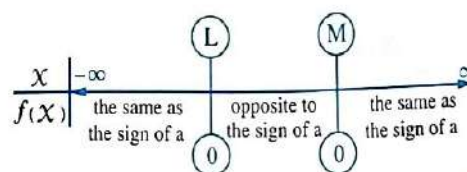
The equation has two real roots, let them be L, M where $L < M$



The sign of the function is as follows:

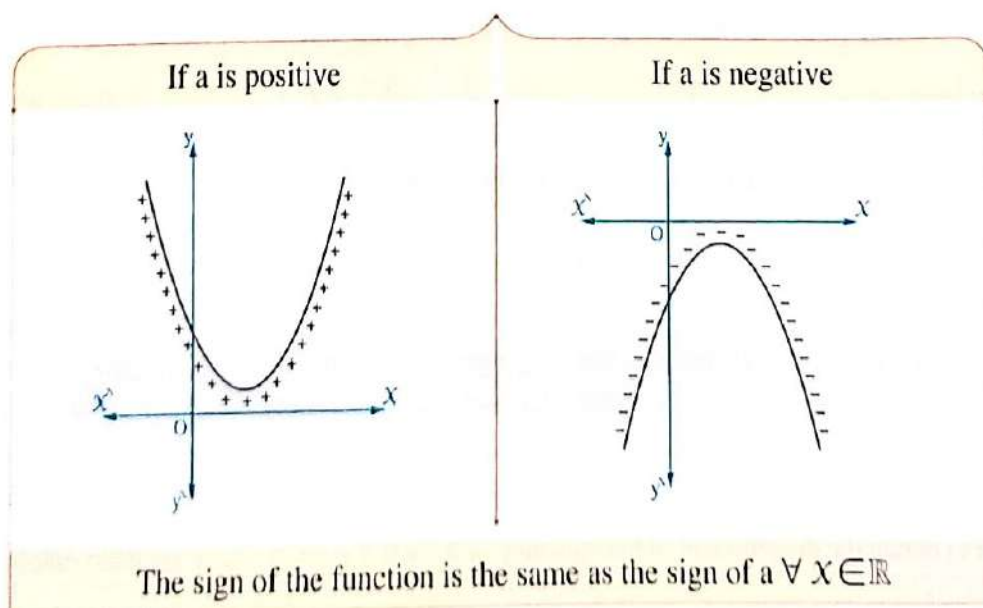
- Is the same as the sign of a at $x \in \mathbb{R} - [L, M]$
- Is opposite to the sign of a at $x \in]L, M[$
- Equals zero at $x \in \{L, M\}$

And we illustrate this on the opposite number line.



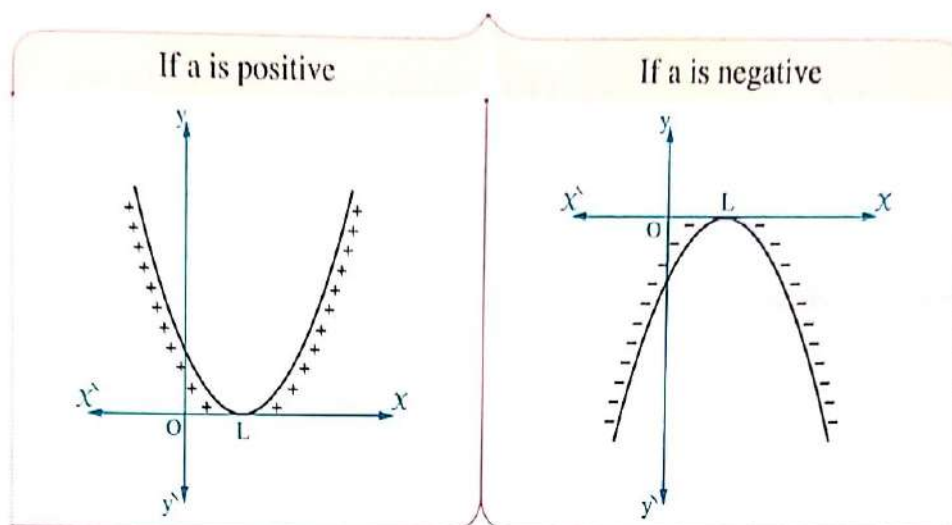
2 The discriminant: $b^2 - 4ac < 0$

There is no real roots for the equation and thus the sign of the function is as follows:



3 The discriminant : $b^2 - 4ac = 0$

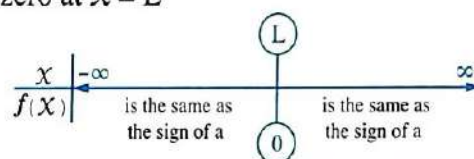
There are two equal roots for the equation, let each of them be L .



The sign of the function is as follows :

- Is the same as a at $X \neq L$
- Is equal to zero at $X = L$

We can illustrate this on the opposite number line.



Example 2

Draw the graph of the function : $f : f(X) = X^2 - 5X + 6$ in the interval $[0, 5]$

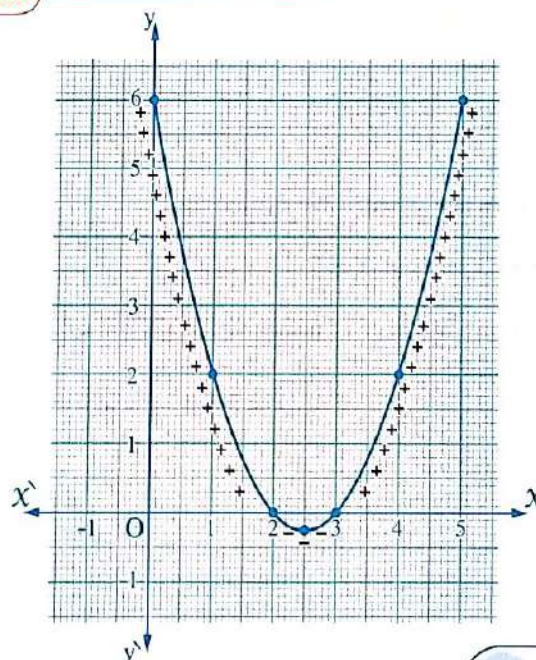
, from the graph determine the sign of the function f in \mathbb{R}

Solution

X	0	1	2	2.5	3	4	5
$f(X)$	6	2	0	-0.25	0	2	6

From the graph, we notice that the sign of f is :

- Positive at $X \in \mathbb{R} - [2, 3]$
- Negative at $X \in]2, 3[$
- $f(X) = 0$ at $X \in \{2, 3\}$



Remark

If the required is investigating the sign of the function in the given interval, then the sign of f is :

- Positive at $x \in [0, 2[\cup]3, 5]$ or $[0, 5] - [2, 3]$
- Negative at $x \in]2, 3[$
- $f(x) = 0$ at $x \in \{2, 3\}$

Remember that

In the previous example :

- The domain of the function f is the set of the real numbers \mathbb{R}
- The range of the function f is $[-0.25, \infty[$
- The vertex of the curve is $(2.5, -0.25)$ and the function has a minimum value equals -0.25
- The symmetry axis equation is $x = 2.5$

Example 3

Draw the graph of the function :

$f : f(x) = -x^2 + 4x - 4$ in the interval $[0, 4]$

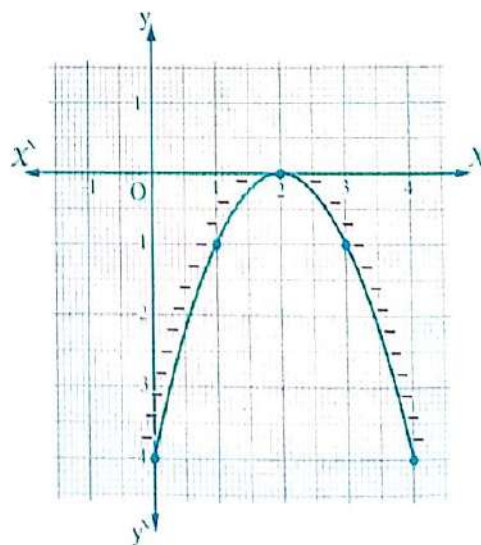
, from the graph determine the sign of the function f in \mathbb{R}

Solution

x	0	1	2	3	4
$f(x)$	-4	-1	0	-1	-4

From the graph, we notice that :

- $f(x) = 0$ at $x = 2$
- The sign of f is negative at $x \in \mathbb{R} - \{2\}$



$\therefore a$ (coefficient of x^2) $= 1 > 0$

$\therefore x = -3$ or $x = 1$

By factorization $\therefore (x+3)(x-1) = 0$

\therefore The equation $x^2 + 2x - 3 = 0$ has two roots.

1 \therefore The discriminant $= b^2 - 4ac = 4 - 4 \times 1 \times (-3) = 4 + 12 = 16 (> \text{zero})$

Solution

3 $f : f(x) = 4x^2 - 12x + 9$

4 $f : f(x) = 9 + 2x - x^2$

1 $f : f(x) = x^2 + 2x - 3$

2 $f : f(x) = x^2 - 3x + 5$

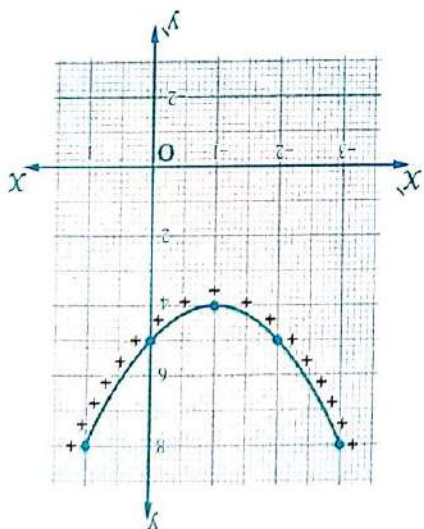
Determine the sign of each of the following functions, showing that on the number line :

Example 5

$f : f(x) = x^2 - 2x - 3$ in the interval $[-2, 4]$, from the graph determine the sign of f in \mathbb{R}

Draw the graph of the function :

TRY TO SOLVE



Function f is positive $\forall x \in \mathbb{R}$

From the graph, we notice that the sign of the

x	$f(x)$
-1	8
0	5
-1	4
-2	5
-3	8

Solution

, from the graph determine the sign of the function f in \mathbb{R}

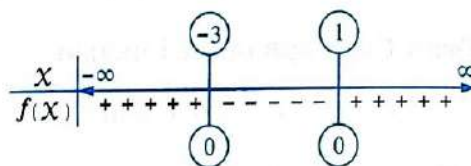
$f : f(x) = x^2 + 2x + 5$ in the interval $[-3, 1]$

Draw the graph of the function :

Example 4

∴ The sign of the function f is :

- positive at $x \in \mathbb{R} - [-3, 1]$
- negative at $x \in]-3, 1[$
- $f(x) = 0$ at $x \in \{-3, 1\}$

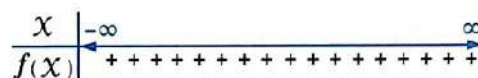


2 ∴ The discriminant $= b^2 - 4ac = 9 - 4 \times 1 \times 5 = 9 - 20 = -11 (< \text{zero})$

∴ The equation : $x^2 - 3x + 5 = 0$ has no real roots

, ∴ $a = 1 > 0$

∴ The sign of the function f is positive $\forall x \in \mathbb{R}$



3 ∴ The discriminant $= b^2 - 4ac = 144 - 4 \times 4 \times 9 = 144 - 144 = 0$

∴ The equation : $4x^2 - 12x + 9 = 0$ has two equal roots

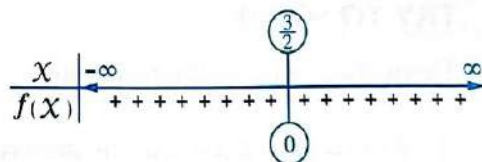
By factorization : ∴ $(2x - 3)^2 = 0$

$$\therefore x = \frac{3}{2}$$

, ∴ $a = 4 > 0$

∴ The sign of the function f is :

- positive at $x \in \mathbb{R} - \left\{ \frac{3}{2} \right\}$
- $f(x) = 0$ at $x = \frac{3}{2}$



4 ∴ The discriminant $= b^2 - 4ac = 4 - 4 \times (-1) \times 9 = 40 (> \text{zero})$

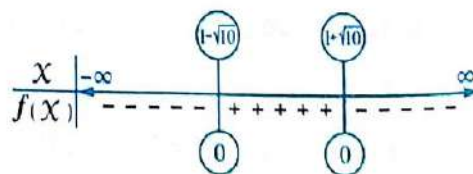
∴ The equation : $9 + 2x - x^2 = 0$ has two roots.

By using the general formula

$$\therefore x = \frac{-2 \pm \sqrt{40}}{-2} = \frac{-2 \pm 2\sqrt{10}}{-2} = 1 \pm \sqrt{10}$$

, ∴ a (coefficient of x^2) $= -1 < 0$ ∴ The sign of the function f is :

- negative at $x \in \mathbb{R} - [1 - \sqrt{10}, 1 + \sqrt{10}]$
- positive at $x \in]1 - \sqrt{10}, 1 + \sqrt{10}[$
- $f(x) = 0$ at $x \in \{1 - \sqrt{10}, 1 + \sqrt{10}\}$



TRY TO SOLVE

Determine the sign of each of the following functions :

1 $f : f(x) = x^2 - x - 6$

2 $f : f(x) = -x^2 - 4x - 4$

3 $f : f(x) = x^2 - 4x + 5$

Example 6

If $f : f(x) = x - 1$, $g : g(x) = x^2 + x - 6$

, find the interval at which the two functions f , g are positive together , also the interval at which f , g are negative together.

Solution

$\therefore f(x) = x - 1$

$\therefore f(x) = 0$ at $x = 1$

, f is positive at $x > 1$

i.e. in the interval $]1, \infty[$

, f is negative at $x < 1$

i.e. in the interval $]-\infty, 1[$

$\therefore g(x) = x^2 + x - 6$,

We get the two roots of the equation $x^2 + x - 6 = 0$ as follows :

$(x - 2)(x + 3) = 0$

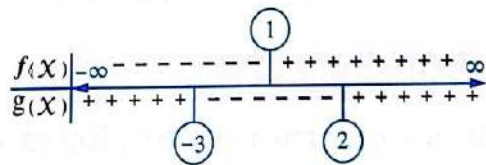
$\therefore x = 2$ or $x = -3$

$\therefore g(x) = 0$ at $x \in \{2, -3\}$

, g is positive at $x \in \mathbb{R} - [-3, 2]$

, g is negative at $x \in]-3, 2[$

By noticing the opposite figure we find :



- f , g are positive together in the interval

$]2, \infty[$ which is the interval representing $]1, \infty[\cap \mathbb{R} - [-3, 2]$

- f , g are negative together at $]-3, 1[$ which is equal to $]-\infty, 1[\cap]-3, 2[$

TRY TO SOLVE

Determine the sign of each of the functions : $f_1 : f_1(x) = 2 - x$ and

$f_2 : f_2(x) = x^2 - 9x + 18$ and when their signs are negative together.

Example 7

Prove that for all values of $X \in \mathbb{R}$ the two roots of the equation : $X^2 + 2kX + k - 2 = 0$ are real and different.

Solution

$$\therefore X^2 + 2kX + k - 2 = 0$$

$$\therefore a = 1, b = 2k, c = k - 2$$

$$\therefore \text{The discriminant} = b^2 - 4ac = (2k)^2 - 4(k - 2) = 4k^2 - 4k + 8$$

and the two roots are real and different if the discriminant is positive ,

thus we investigate the sign of the function

$$f : f(k) = 4k^2 - 4k + 8 \text{ as follows :}$$

$$\therefore \text{The discriminant} = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 8 = 16 - 128 = -112 (< \text{zero})$$

$$\therefore \text{The equation } 4k^2 - 4k + 8 = 0 \text{ has no real roots , } \therefore a > 0$$

$$\therefore \text{The sign of the function } f \text{ is positive for all the values of } k \in \mathbb{R}$$

$$\therefore \text{The discriminant of the equation } X^2 + 2kX + k - 2 = 0 \text{ is positive } \forall X \in \mathbb{R}$$

Thus the two roots of the equation $X^2 + 2kX + k - 2 = 0$ are real and different $\forall X \in \mathbb{R}$

Another solution :

$$\therefore \text{The discriminant of the equation : } X^2 + 2kX + k - 2 = 0 \text{ is } 4k^2 - 4k + 8$$

$$\therefore 4k^2 - 4k + 8 = 4k^2 - 4k + 1 + 7 = (2k - 1)^2 + 7 \text{ is positive } \forall k \in \mathbb{R}$$

$$\therefore \text{The two roots of the equation } X^2 + 2kX + k - 2 = 0 \text{ are real and different } \forall X \in \mathbb{R}$$

Using the Technology

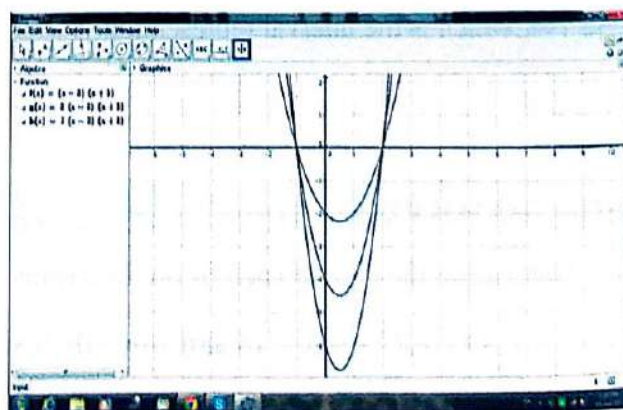
By using the program **GeoGebra** , draw in one graph the functions defined with the following rules :

1 $f(X) = (X - 2)(X + 1)$

2 $g(X) = 2(X - 2)(X + 1)$

3 $k(X) = 3(X - 2)(X + 1)$

You will get the opposite graph.



From the graph , we notice that the three curves are open upwards and intersect the X -axis at the points $(2, 0)$, $(-1, 0)$ and the solution set of each equation which is related to each function is $\{2, -1\}$

- Try to investigate the sign of each of the previous functions.

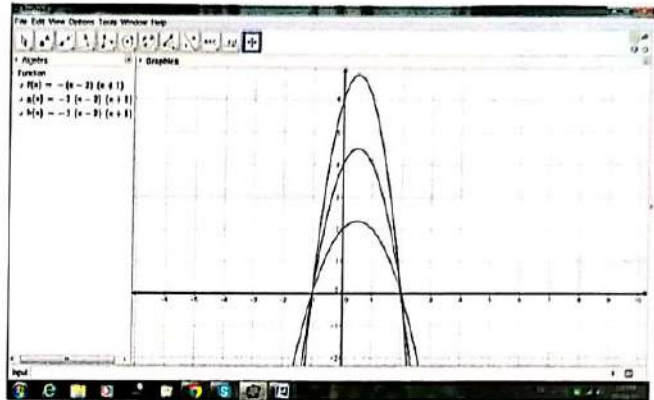
Also, by using the same program draw in one graph the functions defined with the following rules :

1 $f(X) = -(X - 2)(X + 1)$

2 $g(X) = -2(X - 2)(X + 1)$

3 $k(X) = -3(X - 2)(X + 1)$

You will get the opposite graph.



From the graph , we notice that the three curves are open downwards and intersect the X -axis at the previous points $(2, 0)$, $(-1, 0)$, the solution set of each equation which is related to each function is the same solution set $\{2, -1\}$

- Try to investigate the sign of each of the previous functions.

Conclusion

If L , M are the roots of the quadratic equation , then we can form the rule of the function which is related to the quadratic equation on the form :

$$f(X) = a(X - L)(X - M) \text{ where } a \in \mathbb{R} - \{0\}$$

- The curve is open upwards if $a > 0$
- The curve is open downwards if $a < 0$

Lesson

7



Quadratic inequalities in one variable

Preface

- You have studied before inequalities of first degree in one variable as :
 $x + 3 > 5$, $4 - 2x \leq 2$
- Solving an inequality means finding all values of the unknown which satisfy this inequality.
- When solving an inequality in \mathbb{R} , the solution set is an interval.

For example :

When solving the inequality : $-2x + 6 > 10$ in \mathbb{R}

, we find that : $-2x > 4$ $\therefore x < -2$

\therefore The solution set is the real numbers which are less than -2

i.e. The solution set = $]-\infty, -2[$



- In this lesson , you will learn how to solve the inequalities of second degree in one unknown (quadratic inequalities) in \mathbb{R} , there are some examples for these inequalities :

$$x^2 - 5x + 6 > 0 \quad , \quad x^2 + x \geq 2 \quad , \quad x(x - 6) < -5$$

Solving the quadratic inequalities in \mathbb{R}

To solve the quadratic inequality in \mathbb{R} , follow the following steps :

- 1 Write the quadratic function related to the inequality.
- 2 Study the sign of this quadratic function.
- 3 Determine the intervals which satisfy the inequality.

Example 1

Find in \mathbb{R} the solution set of the inequality : $x^2 - 5x + 6 > 0$

Solution

First : Write the quadratic function related to the inequality as follows :

$$f(x) = x^2 - 5x + 6$$

Second : Study the sign of f as follows :

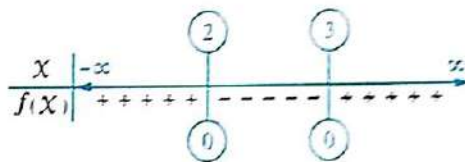
$$\therefore \text{The discriminant} = b^2 - 4ac = 25 - 4 \times 1 \times 6 = 1 (> \text{zero})$$

$$\therefore \text{The equation : } x^2 - 5x + 6 = 0 \text{ has two different roots.}$$

By factorizing :

$$\therefore (x - 2)(x - 3) = 0$$

$$\therefore x = 2 \quad \text{or} \quad x = 3$$



Third : Determine the intervals which satisfy $x^2 - 5x + 6 > 0$ (positive)

$$\therefore \text{The solution set} =]-\infty, 2[\cup]3, \infty[\quad \text{or} \quad \mathbb{R} - [2, 3]$$



Notice that

From the previous example :

The solution set of the inequality : $x^2 - 5x + 6 < 0$ in \mathbb{R} is $]2, 3[$

TRY TO SOLVE

Find in \mathbb{R} the solution set of each of the following inequalities :

1 $x^2 - 2x - 8 > 0$

2 $x^2 - 2x - 8 < 0$

Example 2

Find in \mathbb{R} the solution set of the inequality : $(x + 5)(x - 1) \geq x + 5$

Solution

$$\therefore (x + 5)(x - 1) \geq x + 5 \quad \therefore x^2 + 4x - 5 \geq x + 5 \quad \therefore x^2 + 3x - 10 \geq 0$$

First : Write the quadratic function related to the inequality as follows :

$$f(x) = x^2 + 3x - 10$$

Second : Study the sign of the function f as follows :

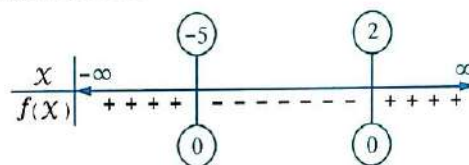
$$\therefore \text{The discriminant} = b^2 - 4ac = 9 - 4 \times 1 \times (-10) = 49 (> \text{zero})$$

$$\therefore \text{The equation } x^2 + 3x - 10 = 0 \text{ has two different roots}$$

By factorizing :

$$\therefore (x - 2)(x + 5) = 0$$

$$\therefore x = 2 \text{ or } x = -5$$



Third : Determine the intervals which satisfy that : $x^2 + 3x - 10 \geq 0$

\therefore The solution set =

$$]-\infty, -5] \cup [2, \infty[\text{ or } \mathbb{R} -]-5, 2[$$



Notice that

From the previous example :

The solution set of the inequality : $(x + 5)(x - 1) \leq x + 5$ in \mathbb{R} is $[-5, 2]$

TRY TO SOLVE

Find in \mathbb{R} the solution set of each of the following inequalities :

1 $2x^2 + 5x \geq 3$

2 $x(x + 6) < 4x + 15$

Example 3

Find in \mathbb{R} the solution set of each of the following inequalities :

1 $x^2 - 3x + 5 < 0$

2 $x^2 + 2x + 4 > 0$

3 $4x - x^2 - 4 < 0$

4 $x^2 - 6x + 9 \leq 0$

Solution

1 By putting $f(x) = x^2 - 3x + 5$ and investigating the sign of the function f , we find that :

$$\text{The discriminant} = b^2 - 4ac = 9 - 4 \times 1 \times 5 = -11 < 0$$

\therefore The equation : $x^2 - 3x + 5 = 0$ has no real roots

$$\therefore a = 1 > 0$$

\therefore The sign of the function f is positive for every $x \in \mathbb{R}$

\therefore The solution set of the inequality : $x^2 - 3x + 5 < 0$ is \emptyset

2 By putting $f(x) = x^2 + 2x + 4$ and investigating the sign of the function f , we find that :

$$\text{The discriminant} = b^2 - 4ac = 4 - 4 \times 1 \times 4 = -12 < 0$$

\therefore The equation : $x^2 + 2x + 4 = 0$ has no real roots

$$\therefore a = 1 > 0$$

\therefore The sign of the function f is positive for every $x \in \mathbb{R}$

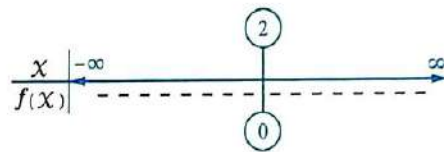
\therefore The solution set of the inequality : $x^2 + 2x + 4 > 0$ is \mathbb{R}

3 By putting $f(x) = 4x - x^2 - 4$ and investigating the sign of f , we find that :

$$\text{The discriminant} = b^2 - 4ac = 16 - 4 \times (-1) \times (-4) = 0$$

\therefore The equation : $4x - x^2 - 4 = 0$ has two equal roots

$$\text{By factorization : } \therefore (x-2)^2 = 0 \quad \therefore x = 2$$



$$\therefore a = -1 < 0$$

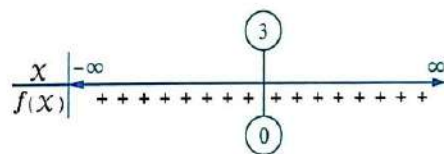
\therefore The function is negative at $x \in \mathbb{R} - \{2\}$, $f(x) = 0$ at $x = 2$

\therefore The solution set of the inequality : $4x - x^2 - 4 < 0$ is $\mathbb{R} - \{2\}$

4 By putting $f(x) = x^2 - 6x + 9$ and investigating the sign of f , we find that :

$$\text{The discriminant} = b^2 - 4ac = 36 - 4 \times 1 \times 9 = 0$$

\therefore The equation : $x^2 - 6x + 9 = 0$ has two equal roots



$$\text{By factorization : } \therefore (x-3)^2 = 0 \quad \therefore x = 3$$

$$\therefore a = 1 > 0$$

\therefore The function is positive at $x \in \mathbb{R} - \{3\}$, $f(x) = 0$ at $x = 3$

\therefore The solution set of the inequality : $x^2 - 6x + 9 \leq 0$ is $\{3\}$

TRY TO SOLVE

Find in \mathbb{R} the solution set of each of the following inequalities :

1 $x^2 + x + 12 > 0$

2 $-x^2 + x - 1 > 0$

3 $x^2 - 2x + 1 > 0$

4 $10x - x^2 - 25 \leq 0$

UNIT
2



Trigonometry

Unit Lessons

Lesson 1 : Directed angle.

Lesson 2 : Systems of measuring angle (Degree measure - radian measure).

Lesson 3 : Trigonometric functions.

Lesson 4 : Related angles.

Lesson 5 : Graphing trigonometric functions.

Lesson 6 : Finding the measure of an angle given the value of one of its trigonometric ratios.

Unit Objectives

By the end of this unit, the student should be able to :

- Recognize the directed angle.
- Recognize the positive measure and negative measure of the directed angle.
- Recognize the standard position of the directed angle.
- Determine the quadrant that the directed angle in its standard position lies.
- Recognize the concept of the equivalent angles.
- Recognize the radian measure of a central angle in a circle.
- Convert a degree measure of an angle into a radian measure and vice versa.
- Find trigonometric functions of some related angles of a special angle.
- Recognize signs of trigonometric functions in each quadrant.
- Use calculator to find trigonometric ratios.
- Use calculator to carry out special arithmetic operations of converting degree measure into radian measure and vice versa.
- Draw trigonometric functions (Sine - Cosine).
- Solve life applications using trigonometric functions.
- Use computer to graph trigonometric functions.
- Find the measure of an angle given one of its trigonometric ratios.

Lesson

1

Directed angle

- We have studied that the angle is the union of two rays with a common starting point.

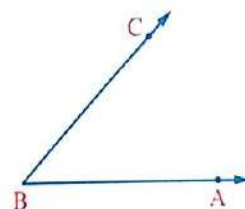
In the opposite figure :

If \overrightarrow{BA} , \overrightarrow{BC} are two rays with a common starting point B, then $\overrightarrow{BA} \cup \overrightarrow{BC} = \angle ABC$ and the two rays \overrightarrow{BA} , \overrightarrow{BC} are called the sides of the angle and the point B is the vertex of the angle.

- As we knew ordering the sides of the angle is not important.

We can write $\angle ABC$ or $\angle CBA$ to express the same angle.

- In this lesson, we will study a new concept which is "directed angle" and some related subjects.



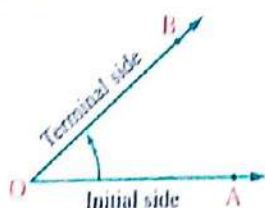
Directed angle

If we take into account the order of the angle sides, such that one of them is the initial side and the other is the terminal side, then the angle is written as "an ordered pair" whose first projection is the initial side and the second projection is the terminal side.

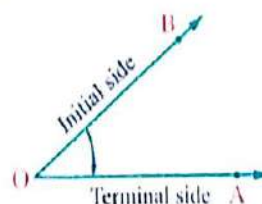
The angle in this case is called "directed angle", its agreed to draw an arrow between its two sides comes out of the initial side to the terminal side.

If \overrightarrow{OA} , \overrightarrow{OB} are the two sides of an angle whose vertex is "O", then :

The ordered pair $(\overrightarrow{OA}, \overrightarrow{OB})$ represents the directed angle $\angle AOB$, whose initial side is \overrightarrow{OA} , and terminal side is \overrightarrow{OB}



The ordered pair $(\overrightarrow{OB}, \overrightarrow{OA})$ represents the directed angle $\angle BOA$ whose initial side is \overrightarrow{OB} , and terminal side is \overrightarrow{OA}



From the previous , we deduce that :

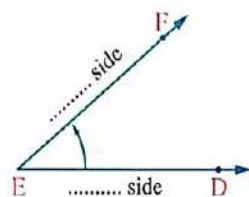
directed angle $\angle AOB \neq$ directed angle $\angle BOA$ because $(\overrightarrow{OA}, \overrightarrow{OB}) \neq (\overrightarrow{OB}, \overrightarrow{OA})$

Definition of the directed angle

The directed angle is an ordered pair of two rays called the sides of the angle with a common starting point called the vertex.

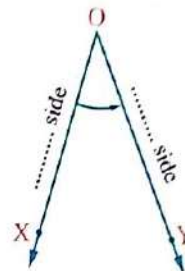
Check your understanding Complete :

1



$(\overrightarrow{ED}, \overrightarrow{EF})$ represents the directed angle $\angle \dots\dots\dots$

2



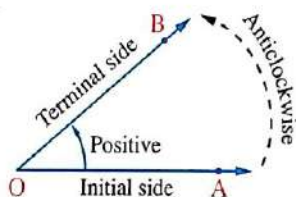
(\dots, \dots) represents the directed angle $\angle XOY$

Positive and negative measures of a directed angle

The measure of the directed angle is

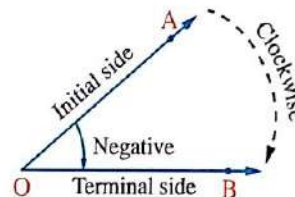
Positive

If the direction of the rotation from the initial side to the terminal side is *anticlockwise*



Negative

If the direction of the rotation from the initial side to the terminal side is *clockwise*

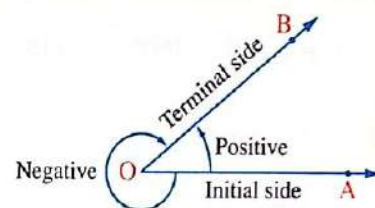


Remark

Each directed angle has two measures, one is positive and the other is negative such that the sum of the absolute values of the two measures equals 360°

i.e. $| \text{Positive measure of the directed angle} |$

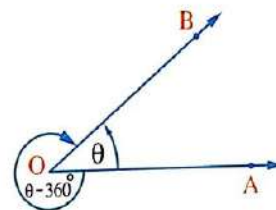
$+ | \text{Negative measure of the same directed angle} | = 360^\circ$



So that :

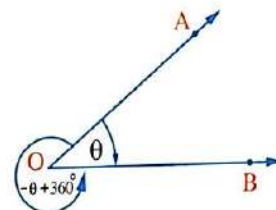
- 1 If the positive measure of the directed angle $= \theta$, then the negative measure of the same directed angle $= \theta - 360^\circ$

For example : The negative measure of the directed angle of measure $210^\circ = 210^\circ - 360^\circ = -150^\circ$



- 2 If the negative measure of the directed angle $= -\theta$, then the positive measure of the same angle $= -\theta + 360^\circ$

For example : The positive measure of the directed angle of measure $(-120^\circ) = -120^\circ + 360^\circ = 240^\circ$



TRY TO SOLVE

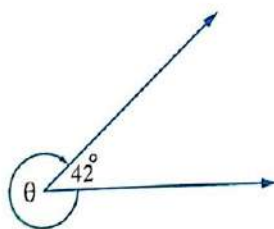
Complete :

- 1 The positive measure of the directed angle whose measure is $(-170^\circ) = \dots\dots\dots$
- 2 The negative measure of the directed angle whose measure is $320^\circ = \dots\dots\dots$

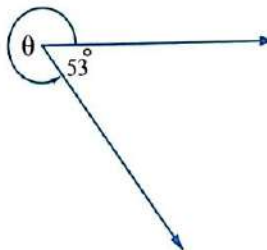
Example 1

Find the measure of the directed angle θ in each of the following figures :

1



2



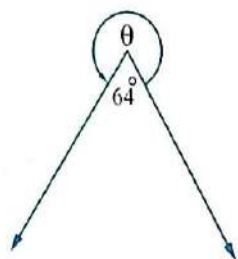
Solution

- 1 \because The rotation direction is clockwise
 \therefore The measure of the angle is negative
 $\therefore \theta = 42^\circ - 360^\circ = -318^\circ$
- 2 \because The rotation direction is anticlockwise
 \therefore The measure of the angle is positive
 $\therefore \theta = -53^\circ + 360^\circ = 307^\circ$

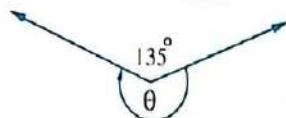
TRY TO SOLVE

Find the measure of the directed angle θ in each of the following figures :

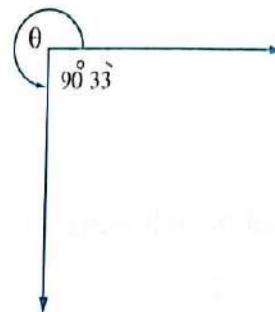
1



2



3

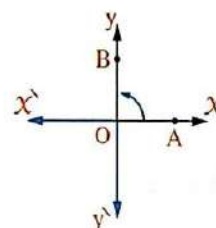
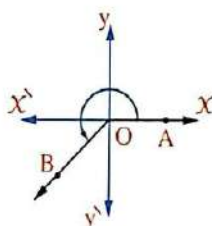
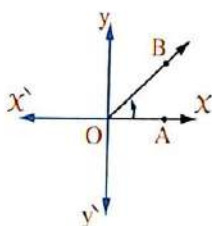
**The standard position of the directed angle**

A directed angle is in the standard position if the following two conditions are satisfied :

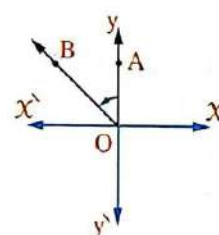
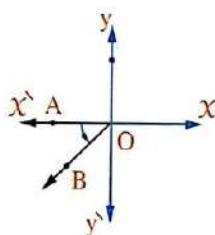
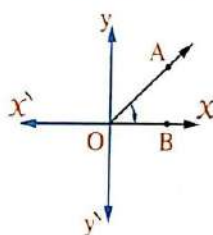
- 1 Its initial side lies on the positive direction of the X -axis.
- 2 Its vertex is the origin point of an orthogonal coordinate plane.

So that :

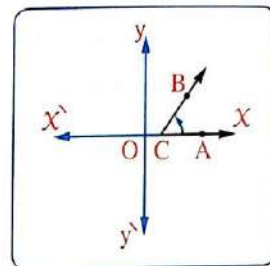
- All the following directed angles are **in the standard position** because they verify the two conditions :



- All the following directed angles are **not in the standard position** because the initial side does not lie on \overrightarrow{OX}



- The directed angle in the opposite figure is **not in the standard position** because its vertex is not the origin point O



Example 2

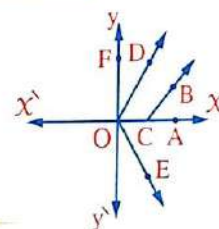
Which of the following ordered pairs represents a directed angle in its standard position ?

1 $(\overrightarrow{OA}, \overrightarrow{OD})$

2 $(\overrightarrow{CA}, \overrightarrow{CB})$

3 $(\overrightarrow{OD}, \overrightarrow{OF})$

4 $(\overrightarrow{OA}, \overrightarrow{OE})$



Solution

- Represents the directed angle $\angle AOD$ in its standard position.
- Represents the directed angle $\angle ACB$ but not in its standard position because its vertex C is not the origin.
- Represents the directed angle $\angle DOF$ but not in its standard position because its initial side \overrightarrow{OD} does not lie on \overrightarrow{OX} .
- Represents the directed angle $\angle AOE$ in its standard position.

TRY TO SOLVE

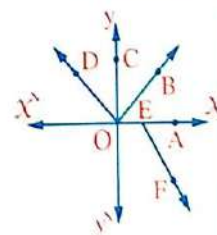
Which of the following ordered pairs represents a directed angle in its standard position ? and why ?

1 $(\overrightarrow{OA}, \overrightarrow{OB})$

2 $(\overrightarrow{EA}, \overrightarrow{EF})$

3 $(\overrightarrow{OC}, \overrightarrow{OA})$

4 $(\overrightarrow{OB}, \overrightarrow{OD})$



Equivalent angles

- If we notice the directed angles in the standard position in the following figures :

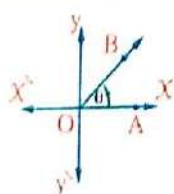


Fig. (1)

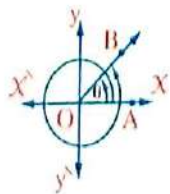


Fig. (2)

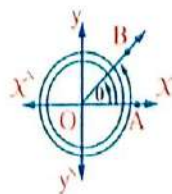


Fig. (3)

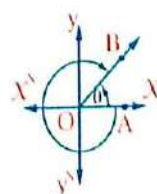


Fig. (4)

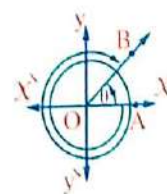


Fig. (5)

We notice the following :

- 1 The angles in the five figures have the same terminal side \overrightarrow{OB}
- 2 The measure of the angle in fig. (1) = θ ,
 The measure of the angle in fig. (2) = $\theta + 360^\circ$,
 The measure of the angle in fig. (3) = $\theta + 2 \times 360^\circ$,
 The measure of the angle in fig. (4) = $-(360^\circ - \theta) = \theta - 360^\circ$
 The measure of the angle in fig. (5) = $-(2 \times 360^\circ - \theta) = \theta - 2 \times 360^\circ$

From this , we can conclude :

If θ is the measure of a directed angle in the standard position, then the angles whose measures are :

$(\theta \pm 360^\circ)$, $(\theta \pm 2 \times 360^\circ)$, $(\theta \pm 3 \times 360^\circ)$... , $(\theta \pm n \times 360^\circ)$, such that n is an integer have common terminal side.

These angles that have common terminal side are called "equivalent angles".

Definition of the equivalent angles

Several directed angles in the standard position are said to be equivalent when they have one common terminal side.

Example 3

Determine two angles , one with positive measure and the other with negative measure having common terminal side for :

1 100°

2 -250°

Solution

1 An angle with positive measure = $100^\circ + 360^\circ = 460^\circ$

An angle with negative measure = $100^\circ - 360^\circ = -260^\circ$

2 An angle with positive measure = $-250^\circ + 360^\circ = 110^\circ$

An angle with negative measure = $-250^\circ - 360^\circ = -610^\circ$

Notice that

There are an infinite number of other positive and negative measures of angles having common terminal side.

Example 4

Determine the smallest positive measure for each of the angles whose measures are as follows :

- 1 -62° 2 -225° 3 530° 4 -790°

Solution

- 1 The smallest positive measure $= -62^\circ + 360^\circ = 298^\circ$
 2 The smallest positive measure $= -225^\circ + 360^\circ = 135^\circ$
 3 The smallest positive measure $= 530^\circ - 360^\circ = 170^\circ$
 4 The smallest positive measure $= -790^\circ + 3 \times 360^\circ = 290^\circ$

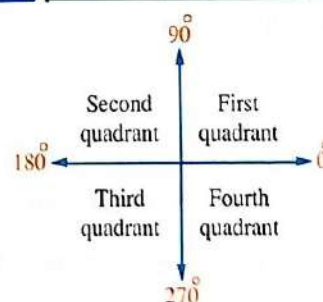
TRY TO SOLVE

- 1 Determine a negative measure for each of :
 (1) 72° (2) 1150°
 2 Determine the smallest positive measure for each of :
 (1) -115° (2) 405°

Angle position in the orthogonal coordinate plane

We know that the orthogonal coordinate plane is divided into four quadrants as in the opposite figure.

The position of the directed angle is determined by its terminal side when it is in its standard position.



If we draw the directed angle $\angle AOB$ in the standard position of positive measure θ , then :

The terminal side \overrightarrow{OB} lies in a quadrant as follows :

First quadrant	Second quadrant	Third quadrant	Fourth quadrant
$\angle AOB$ lies in the first quadrant $0^\circ < \theta < 90^\circ$	$\angle AOB$ lies in the second quadrant $90^\circ < \theta < 180^\circ$	$\angle AOB$ lies in the third quadrant $180^\circ < \theta < 270^\circ$	$\angle AOB$ lies in the fourth quadrant $270^\circ < \theta < 360^\circ$

Remark

If the terminal side lies on one of the two axes, then the angle is called "quadrantal angle".

i.e. The angles whose measures are 0° , 90° , 180° , 270° , 360° are quadrantal angles.

Example 5

Determine the quadrant in which each of the directed angles whose measures are as follows lies :

- | | | | |
|---------------|---------------|-----------------|---------------|
| 1 213° | 2 132° | 3 -310° | 4 -12° |
| 5 270° | 6 964° | 7 -1070° | |

Solution

1 $\because 180^\circ < 213^\circ < 270^\circ \quad \therefore$ The angle lies in the third quadrant.

2 $\because 90^\circ < 132^\circ < 180^\circ \quad \therefore$ The angle lies in the second quadrant.

3 The smallest positive measure $= -310^\circ + 360^\circ = 50^\circ$

$$\because 0^\circ < 50^\circ < 90^\circ$$

\therefore The angle of measure 50° lies in the first quadrant

\therefore The angle of measure -310° also lies in the first quadrant.

Notice that

To determine the quadrant which the directed angle lies in, we have to find the smallest positive measure of it.

- 4 The smallest positive measure $= -12^\circ + 360^\circ = 348^\circ$
 $\therefore 270^\circ < 348^\circ < 360^\circ$
 \therefore The angle of measure 348° lies in the fourth quadrant.
 \therefore The angle of measure -12° also lies in the fourth quadrant.
- 5 270° is a quadrantal angle.
- 6 The smallest positive measure $= 964^\circ - 2 \times 360^\circ = 244^\circ$
 $\therefore 180^\circ < 244^\circ < 270^\circ$
 \therefore The angle of measure 244° lies in the third quadrant.
 \therefore The angle of measure 964° also lies in the third quadrant.
- 7 The smallest positive measure $= -1070^\circ + 3 \times 360^\circ = 10^\circ$
 $\therefore 0^\circ < 10^\circ < 90^\circ$
 \therefore The angle of measure 10° lies in the first quadrant.
 \therefore The angle of measure -1070° also lies in the first quadrant.

TRY TO SOLVE

Determine the quadrant in which each of the directed angles whose measures are as follows lies :

1 67°

2 -67°

3 220°

4 -220°

5 875°

6 -2020°

Lesson

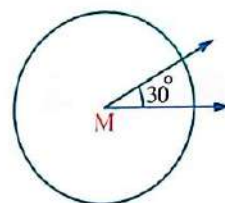
2



Systems of measuring angle (Degree measure - Radian measure)

Degree measure system

It depends on dividing the circle into 360 equal arcs in length, then the central angle whose sides pass through the two ends of one of the arcs, its measure equals one degree which is symbolized by 1° , and the central angle which subtends between its sides 30 arcs of this arcs, its measure equals 30° and so on.



The unit of measurement of the degree measure

The degree is the unit of measuring the angle in the degree measure which is divided into 60 equal parts, each part is called a minute, and it is symbolized by $1'$, also the minute is divided into 60 equal parts, each part is called a second and it is symbolized by $1''$.

i.e. $1^\circ = 60'$, $1' = 60''$

In this type of measuring angle, the protractor is used as an instrument for measuring angles in degrees.

Remember that

Calculator can be used to convert parts of degrees and minutes into minutes and seconds and vice versa

Such as

$$* 37 \frac{3}{8}^\circ = 37^\circ 22' 30''$$

$$37 \frac{3}{8} \text{ [0.] [.] [=]} 37^\circ 22' 30''$$

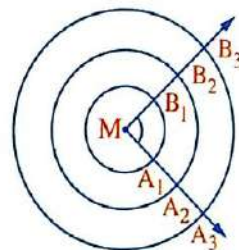
$$* 70^\circ 37' 30'' = 70 \frac{5}{8}^\circ$$

$$70 \text{ [0.] [.] [0.] [.] [=]} \text{ [SHIFT] [S↔D]} 70 \frac{5}{8}$$

Radian measure system

This measure depends on the following geometrical fact :

In the concentric circles , the ratio of the length of the arc of any central angle, and the length of the radius of its corresponding circle equals constant quantity.



i.e.
$$\frac{\text{length of } \widehat{A_1 B_1}}{MA_1} = \frac{\text{length of } \widehat{A_2 B_2}}{MA_2} = \frac{\text{length of } \widehat{A_3 B_3}}{MA_3} = \text{constant quantity}$$

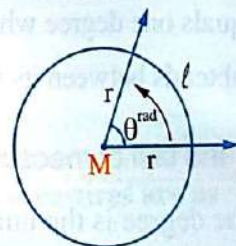
and this constant is the radian measure of the angle.

i.e.
$$\begin{aligned} &\text{The radian measure of a central angle in a circle} \\ &= \frac{\text{length of the arc which the central angle subtends}}{\text{length of the radius of this circle}} \end{aligned}$$

Definition

If θ^{rad} is the radian measure of a central angle in a circle of radius length r subtends an arc of length ℓ , then

$$\theta^{\text{rad}} = \frac{\ell}{r}$$



and since the radius length of the circle r is constant , then the radian measure of the central angle varies directly as the length of the subtended arc.

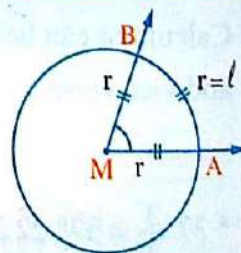
The unit of measurement of the radian measure

The radian angle is the unit of measuring the angle in the radian measure , and we can define the radian angle as follows which is denoted by (1^{rad}) and is read as one radian.

Definition

The radian angle is a central angle in a circle subtends an arc of length equals the length of the radius of the circle.

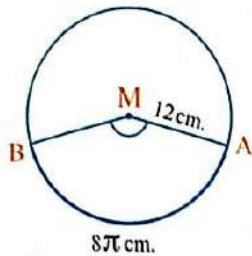
Notice : $\theta^{\text{rad}} = \frac{\ell}{r} \quad \therefore \theta^{\text{rad}} = \frac{r}{r} = 1^{\text{rad}}$



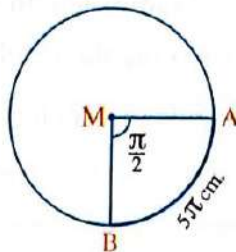
For example : The measure of the central angle that subtends an arc whose length equals double the length of the radius of its circle $= 2^{\text{rad}}$

Example 1

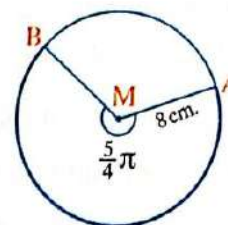
In each of the following circles, find the required under each figure approximating to the nearest tenth :

1

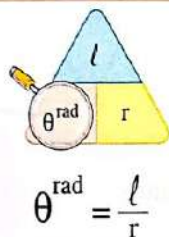
Find : $m(\angle AMB)$ in radian measure.

2

Find : the radius length of circle M

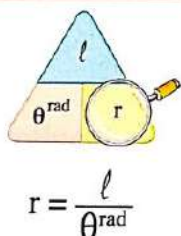
3

Find : the length of \widehat{AB} the greater.

Solution**1**

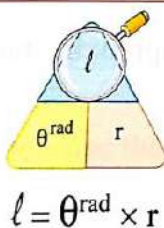
$$\theta^{\text{rad}} = ? , l = 8\pi \text{ cm.} , r = 12 \text{ cm.}$$

$$\begin{aligned} \therefore m(\angle AMB) \text{ in radian measure} &= \frac{l}{r} = \frac{8\pi}{12} \\ &= \frac{2}{3}\pi \approx 2.1^{\text{rad}} \end{aligned}$$

2

$$r = ? , l = 5\pi \text{ cm.} , \theta^{\text{rad}} = \frac{\pi}{2}$$

$$\begin{aligned} \therefore \text{The radius length} &= \frac{l}{\theta^{\text{rad}}} = \frac{5\pi}{\frac{\pi}{2}} \\ &= 5\pi \times \frac{2}{\pi} = 10 \text{ cm.} \end{aligned}$$

3

$$l = ? , \theta^{\text{rad}} = \frac{5}{4}\pi , r = 8 \text{ cm.}$$

$$\begin{aligned} \therefore \text{The length of } \widehat{AB} \text{ the greater} &= \theta^{\text{rad}} \times r \\ &= \frac{5}{4}\pi \times 8 = 10\pi \approx 31.4 \text{ cm.} \end{aligned}$$

Remark

If the length of the radius of a circle is the unit, then the circle is called "the unit circle", where $\theta^{\text{rad}} = l$

For example : In the unit circle, the central angle that subtends an arc of length $\frac{1}{2}\pi$ unit length has a radian measure $= \frac{1}{2}\pi \approx 1.57^{\text{rad}}$

TRY TO SOLVE

- 1 Find the radian measure of the central angle which subtends an arc of length 15 cm. if the radius length of the circle is 10 cm.
- 2 Find the length of the arc in a circle of radius length 8 cm. if the measure of the central angle subtended by it is $\frac{7\pi}{12}$ approximating the result to the nearest hundredth.
- 3 Find the length of the radius of the circle in which a central angle of measure $\frac{9\pi}{8}$ is drawn subtending an arc of length 24 cm. to the nearest tenth.

The relation between the radian measure and the degree measure

You have known that, in a circle : $\frac{\text{Measure of the arc}}{\text{Measure of the circle}} = \frac{\text{Length of this arc}}{\text{Circumference of the circle}}$

i.e. In the opposite figure : $\frac{m(\widehat{AB})}{360^\circ} = \frac{\text{Length of } \widehat{AB}}{2\pi r}$

$$\therefore m(\angle AMB) = m(\widehat{AB}) \quad \therefore \frac{m(\angle AMB)}{180^\circ} = \frac{\text{Length of } \widehat{AB}}{\pi r}$$



Assuming that : $m(\angle AMB)$ equals x° in degrees and equals θ^{rad} in radians and the length of $\widehat{AB} = l$

$$\therefore \frac{x^\circ}{180^\circ} = \frac{l}{\pi r} \quad , \quad \therefore \theta^{\text{rad}} = \frac{l}{r}$$

$$\therefore \frac{x^\circ}{180^\circ} = \frac{\theta^{\text{rad}}}{\pi} \quad \text{and from it} \quad \theta^{\text{rad}} = x^\circ \times \frac{\pi}{180^\circ} \quad , \quad x^\circ = \theta^{\text{rad}} \times \frac{180^\circ}{\pi}$$

Example 2

- 1 Find the radian measure of the angle whose degree measure is $75^\circ 32' 15''$ approximating the result to the nearest thousandth.
- 2 Find the degree measure of the angle whose radian measure is 2.38^{rad}

Solution

$$1 \quad \therefore \theta^{\text{rad}} = x^\circ \times \frac{\pi}{180^\circ}$$

$$\therefore \theta^{\text{rad}} = 75^\circ 32' 15'' \times \frac{\pi}{180^\circ} \approx 1.318^{\text{rad}}$$

$$2 \quad \therefore x^\circ = \theta^{\text{rad}} \times \frac{180^\circ}{\pi}$$

$$\therefore x^\circ = 2.38^{\text{rad}} \times \frac{180^\circ}{\pi} \approx 136^\circ 21' 50''$$

Enrichment Information

There is another unit of measuring angles called (Grad) which equals $\frac{1}{200}$ of the measure of the straight angle.

If x , θ , y are the measures of three angles respectively in degrees, radian and grade

$$\text{, then } \frac{x^\circ}{180^\circ} = \frac{\theta^{\text{rad}}}{\pi} = \frac{y^{\text{grad}}}{200}$$

TRY TO SOLVE

- 1 Convert the measure of the angle 1.2^{rad} into degrees.
- 2 Convert the measure of the angle $72^{\circ} 30'$ into radians approximating the result to the nearest hundredth.

Remarks

- 1 If the radian measure of an angle equals π (radian), then its degree measure

$$= \pi \times \frac{180^{\circ}}{\pi} = 180^{\circ}$$

i.e. π in radians is equivalent to 180° in degrees.

For example : $\frac{3}{5} \pi$ is equivalent to $\frac{3}{5} \times 180^{\circ} = 108^{\circ}$

- 2 If the degree measure of an angle is known, and it is required to convert it into radian measure in terms of π , then we use the relation : $\theta^{\text{rad}} = x^{\circ} \times \frac{\pi}{180^{\circ}}$ without substituting with π

For example : • 18° is equivalent to $18^{\circ} \times \frac{\pi}{180^{\circ}} = \frac{\pi}{10}$

• 135° is equivalent to $135^{\circ} \times \frac{\pi}{180^{\circ}} = \frac{3}{4} \pi$

Example 3

Determine the quadrant in which the directed angle of each of the angles whose measures are as follows lies :

1 2.02^{rad}

2 -7.3^{rad}

3 $\frac{5}{4} \pi$

Solution

To determine the quadrant in which the directed angle lies, we find its degree measure :

1 $\because x^{\circ} = \theta^{\text{rad}} \times \frac{180^{\circ}}{\pi} = 2.02 \times \frac{180^{\circ}}{\pi} \approx 115^{\circ} 44' 15''$

\therefore The angle whose measure is 2.02^{rad} is equivalent to $115^{\circ} 44' 15''$ in degrees.

\therefore The angle of measure $115^{\circ} 44' 15''$ lies in the second quadrant

\therefore The angle of measure 2.02^{rad} lies in the second quadrant.

2 $\because x^{\circ} = -7.3^{\text{rad}} \times \frac{180^{\circ}}{\pi} \approx -418^{\circ} 15' 33''$

\therefore The angle of measure $-418^{\circ} 15' 33''$ is equivalent to

$$-418^{\circ} 15' 33'' + 2 \times 360^{\circ} = 301^{\circ} 44' 27''$$

\therefore The angle of measure $301^\circ 44' 27''$ lies in the fourth quadrant

\therefore The angle of measure -7.3^{rad} lies in the fourth quadrant.

3 $\therefore \frac{5\pi}{4}$ is equivalent to $\frac{5}{4} \times 180^\circ = 225^\circ$

\therefore The angle whose measure is 225° lies in the third quadrant.

\therefore The angle whose measure is $\frac{5\pi}{4}$ lies in the third quadrant.

Remark

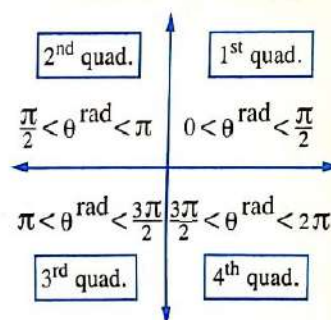
It is possible to determine the quadrant in which the directed angle - whose radian measure is known in terms of π - lies without converting to degrees using the opposite figure :

For example :

By using the opposite figure we can determine in which quadrant the angle whose measure is $\frac{5}{4}\pi$ in the last example lies where

$$\pi < \frac{5\pi}{4} < \frac{3\pi}{2}$$

\therefore The angle whose measure is $\frac{5}{4}\pi$ lies in the third quadrant.



TRY TO SOLVE

Complete :

- 1 The angle of measure $\frac{5\pi}{3}$ lies in the quadrant.
- 2 The angle of measure -0.3π lies in the quadrant.
- 3 The angle of measure 5.7^{rad} lies in the quadrant.
- 4 The angle of measure -6.4^{rad} lies in the quadrant.

Example 4

Find the length of the arc subtended by the central angle whose measure is $152^\circ 26' 17''$ drawn in a circle of radius length 10.5 cm. approximating the result to the nearest cm.

Solution

$$\therefore \theta^{\text{rad}} = x^\circ \times \frac{\pi}{180^\circ} = 152^\circ 26' 17'' \times \frac{\pi}{180^\circ} \approx 2.6605^{\text{rad}}$$

$$\therefore l = \theta^{\text{rad}} \times r = 2.6605 \times 10.5 \approx 28 \text{ cm.}$$

Example 5

Find each of the radian measure and the degree measure of the central angle subtending an arc of length 12.6 cm. in a circle of radius length 7.2 cm.

Solution

$$\theta^{\text{rad}} = \frac{l}{r} = \frac{12.6}{7.2} = 1.75^{\text{rad}}$$

$$\therefore x^{\circ} = 1.75^{\text{rad}} \times \frac{180^{\circ}}{\pi} \approx 100^{\circ} 16' 3''$$

Example 6

Find the circumference of the circle that has an inscribed angle of measure 30° subtending an arc of length 5 cm.

Solution

\therefore Measure of the inscribed angle = 30°

\therefore Measure of the corresponding central angle = 60°

$$\therefore \theta^{\text{rad}} = 60^{\circ} \times \frac{\pi}{180^{\circ}} = \frac{\pi}{3} \quad \therefore r = \frac{l}{\theta^{\text{rad}}} = 5 \div \left(\frac{\pi}{3}\right) = \frac{15}{\pi} \text{ cm.}$$

$$\therefore \text{Circumference of the circle} = 2\pi r = 2\pi \times \frac{15}{\pi} = 30 \text{ cm.}$$

Example 7

Two angles, the sum of their radian measures = $3\frac{1}{7}^{\text{rad}}$, and the difference between their degree measures = 30° , find the measure of each of them in degrees and in radians.

$$\left(\pi = \frac{22}{7}\right)$$

Solution

$$\therefore 3\frac{1}{7}^{\text{rad}} = \frac{22}{7} \times \frac{180^{\circ}}{\pi} = 180^{\circ} \text{ assuming the two angles are } A, B \text{ such that : } m(\angle A) > m(\angle B)$$

$$\therefore m(\angle A) + m(\angle B) = 180^{\circ}, m(\angle A) - m(\angle B) = 30^{\circ}$$

By adding :

$$\therefore 2m(\angle A) = 210^{\circ}$$

$$\therefore m(\angle A) = 105^{\circ}$$

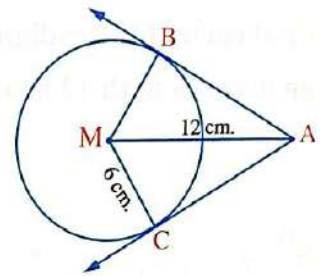
$$\therefore m(\angle B) = 75^{\circ}$$

$$\therefore m(\angle A) \text{ in radians} = 105^{\circ} \times \frac{\pi}{180^{\circ}} \approx 1.83^{\text{rad}}$$

$$\therefore m(\angle B) \text{ in radians} = 75^{\circ} \times \frac{\pi}{180^{\circ}} \approx 1.31^{\text{rad}}$$

Example 8

In the opposite figure : \overrightarrow{AB} , \overrightarrow{AC} are two tangents to the circle M whose radius length is 6 cm. If $AM = 12$ cm.
find the length of the major arc \widehat{BC} to the nearest integer.



Solution

$\therefore \overrightarrow{AC}$ is a tangent to the circle M

$\therefore \overline{MC} \perp \overline{AC}$

In $\triangle AMC$:

$$\therefore m(\angle ACM) = 90^\circ, MC = \frac{1}{2} AM$$

$$\therefore m(\angle CAM) = 30^\circ$$

$$\therefore m(\angle AMC) = 60^\circ$$

$\therefore \overrightarrow{MA}$ bisects $\angle BMC$

$$\therefore m(\angle BMC) = 120^\circ$$

$$\therefore m(\angle BMC) \text{ the reflex} = 360^\circ - 120^\circ = 240^\circ$$

$$\therefore \theta^{\text{rad}} = x^\circ \times \frac{\pi}{180^\circ}$$

$$\therefore \theta^{\text{rad}} = 240^\circ \times \frac{\pi}{180^\circ} = \frac{4\pi}{3}$$

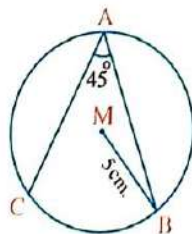
$$\therefore l = \theta^{\text{rad}} \times r$$

$$\therefore \text{The length of } \widehat{BC} \text{ the major} = \frac{4\pi}{3} \times 6 = 8\pi \approx 25 \text{ cm.}$$

TRY TO SOLVE

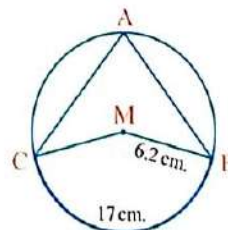
Complete :

1



The length of \widehat{BC} = cm.

2



$m(\angle A)$ = $^\circ$

Lesson

3

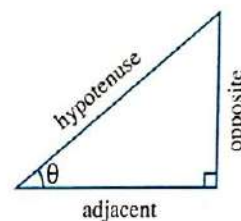
Trigonometric functions

We have studied before the basic trigonometric ratios of an acute angle and we have known that :

In any right-angled triangle :

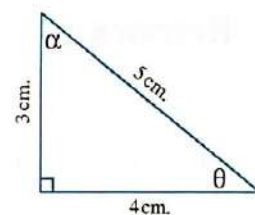
$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad , \quad \cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$, \quad \tan \theta = \frac{\text{Opposite}}{\text{Adjacent}}$$



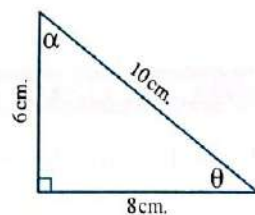
In the opposite figure :

$\sin \theta = \frac{3}{5}$	$\cos \theta = \frac{4}{5}$	$\tan \theta = \frac{3}{4}$
$\sin \alpha = \frac{4}{5}$	$\cos \alpha = \frac{3}{5}$	$\tan \alpha = \frac{4}{3}$



and if we draw another triangle similar to the previous triangle , we find that :

$\sin \theta = \frac{6}{10} = \frac{3}{5}$	$\cos \theta = \frac{8}{10} = \frac{4}{5}$	$\tan \theta = \frac{6}{8} = \frac{3}{4}$
$\sin \alpha = \frac{8}{10} = \frac{4}{5}$	$\cos \alpha = \frac{6}{10} = \frac{3}{5}$	$\tan \alpha = \frac{8}{6} = \frac{4}{3}$



From the previous , we deduce that :

1 $\sin \theta$, $\cos \theta$, $\tan \theta$ in the two triangles are equal.

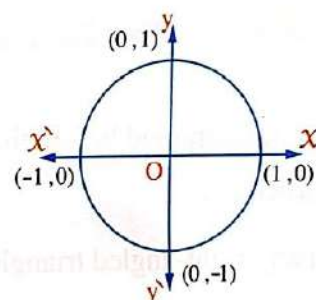
i.e. The trigonometric ratio of the angle is constant and does not depend on the area of the triangle.

2 $\sin \theta \neq \sin \alpha$, $\cos \theta \neq \cos \alpha$, $\tan \theta \neq \tan \alpha$ in any of the two triangles.

i.e. The trigonometric ratio is changed by the change of the angle which is known by "The trigonometric functions"

The unit circle

In the orthogonal coordinate system , the circle of centre at the origin point and of radius equals the unit of length is called a unit circle.



Notice from the previous figure :

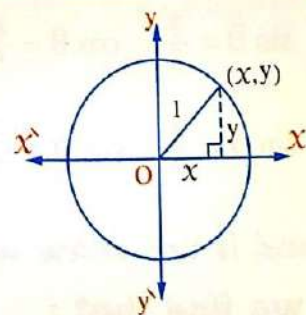
- The unit circle intersects the x -axis at two points which are $(1, 0)$, $(-1, 0)$
- The unit circle intersects the y -axis at two points which are $(0, 1)$, $(0, -1)$

Remark

If the point $(x, y) \in$ the unit circle , then

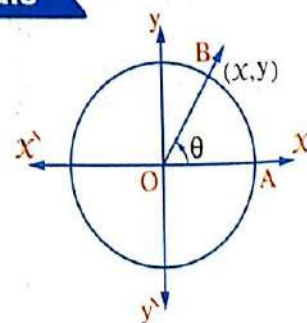
* $x^2 + y^2 = 1$ from Pythagoras' theorem.

* $x \in [-1, 1]$, $y \in [-1, 1]$



The basic trigonometric functions and their reciprocals

If we draw the directed angle AOB in the standard position and its terminal side intersects the unit circle at the point B (x, y) and if $m(\angle AOB) = \theta$, then



First The basic trigonometric functions of the angle of measure θ are :

- 1 Cosine of the angle = x - coordinate of the point B **i.e.** $\cos \theta = x$
- 2 Sine of the angle = y - coordinate of the point B **i.e.** $\sin \theta = y$
- 3 Tangent of the angle = $\frac{y - \text{coordinate of the point B}}{x - \text{coordinate of the point B}}$ **i.e.** $\tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$, where $x \neq 0$

Notice that The coordinates of the point B (x, y) can be written as $(\cos \theta, \sin \theta)$

Second The reciprocals of the basic trigonometric functions for the angle of measure θ are :

- 1 The secant of the angle (\sec) = $\frac{1}{x - \text{coordinate of the point B}}$
i.e. $\sec \theta = \frac{1}{x} = \frac{1}{\cos \theta}$, where $x \neq 0$
- 2 The cosecant of the angle (\csc) = $\frac{1}{y - \text{coordinate of the point B}}$
i.e. $\csc \theta = \frac{1}{y} = \frac{1}{\sin \theta}$, where $y \neq 0$
- 3 The cotangent of the angle (\cot) = $\frac{x - \text{coordinate of the point B}}{y - \text{coordinate of the point B}}$
i.e. $\cot \theta = \frac{x}{y} = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}$, where $y \neq 0$

Example 1

Find all trigonometric functions for an angle of measure θ which is drawn in the standard position and its terminal side intersects the unit circle at the point A in each of the following :

1 A $\left(\frac{3}{5}, \frac{4}{5}\right)$

2 A $(-1, 0)$

3 A $\left(-\frac{1}{2}, y\right)$, where $y > 0$

4 A $(-x, x)$ where $x > 0$

Solution

$$1 \quad \cos \theta = \frac{3}{5}, \quad \sin \theta = \frac{4}{5}, \quad \tan \theta = \frac{4}{5} \div \frac{3}{5} = \frac{4}{3}$$

$$, \sec \theta = \frac{5}{3}, \quad \csc \theta = \frac{5}{4}, \quad \cot \theta = \frac{3}{4}$$

$$2 \quad \cos \theta = -1, \quad \sin \theta = 0, \quad \tan \theta = \frac{0}{-1} = 0$$

$$, \sec \theta = -1, \quad \csc \theta = \frac{1}{0} \text{ (undefined)}, \quad \cot \theta = \frac{-1}{0} \text{ (undefined)}$$

$$3 \quad \because x^2 + y^2 = 1 \quad \therefore \left(-\frac{1}{2}\right)^2 + y^2 = 1$$

$$\therefore y^2 = 1 - \frac{1}{4} = \frac{3}{4} \quad \therefore y = \pm \frac{\sqrt{3}}{2}$$

$$, \because y > 0 \quad \therefore y = \frac{\sqrt{3}}{2} \quad \therefore A\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\therefore \cos \theta = -\frac{1}{2}, \quad \sin \theta = \frac{\sqrt{3}}{2}, \quad \tan \theta = \frac{\sqrt{3}}{2} \div -\frac{1}{2} = -\sqrt{3}$$

$$, \sec \theta = -2, \quad \csc \theta = \frac{2}{\sqrt{3}}, \quad \cot \theta = \frac{-1}{\sqrt{3}}$$

$$4 \quad \because x^2 + y^2 = 1 \quad \therefore (-x)^2 + x^2 = 1$$

$$\therefore 2x^2 = 1 \quad \therefore x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}} \quad , \because x > 0$$

$$\therefore x = \frac{1}{\sqrt{2}} \quad \therefore A\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\therefore \cos \theta = \frac{1}{\sqrt{2}}, \quad \sin \theta = \frac{1}{\sqrt{2}}, \quad \tan \theta = \frac{1}{\sqrt{2}} \div \frac{1}{\sqrt{2}} = 1$$

$$, \sec \theta = \sqrt{2}, \quad \csc \theta = \sqrt{2}, \quad \cot \theta = 1$$

TRY TO SOLVE

Find all trigonometric functions of an angle θ drawn in the standard position whose terminal side intersects the unit circle at the point B for each of the following :

$$1 \quad B\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$2 \quad B(0, x), \text{ where } x < 0$$

$$3 \quad B(-y, -y), \text{ where } y > 0$$

Remark

The equivalent angles have the same trigonometric functions :

i.e. For all values of $n \in \mathbb{Z}$ (set of integers) , then

- $\cos(\theta + 2n\pi) = \cos \theta = x$, $\sec(\theta + 2n\pi) = \sec \theta = \frac{1}{x}$, where $x \neq 0$
- $\sin(\theta + 2n\pi) = \sin \theta = y$, $\csc(\theta + 2n\pi) = \csc \theta = \frac{1}{y}$, where $y \neq 0$
- $\tan(\theta + 2n\pi) = \tan \theta = \frac{y}{x}$, where $x \neq 0$, $\cot(\theta + 2n\pi) = \cot \theta = \frac{x}{y}$, where $y \neq 0$

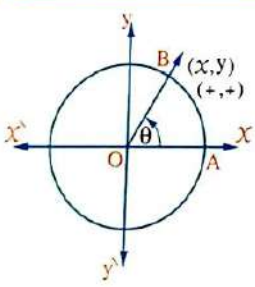
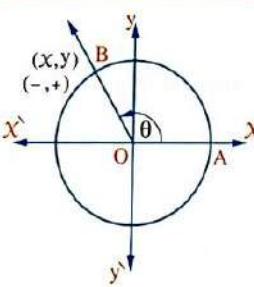
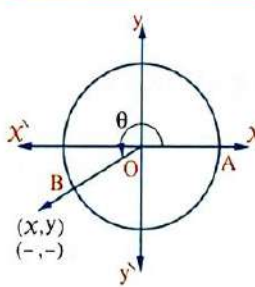
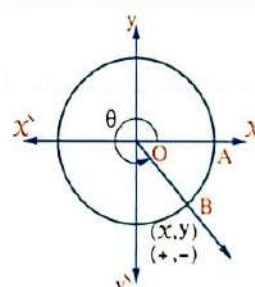
For example :

- $\cos 420^\circ = \cos(60^\circ + 360^\circ) = \cos 60^\circ$
- $\sec 840^\circ = \sec(120^\circ + 2 \times 360^\circ) = \sec 120^\circ$
- $\tan(-1500^\circ) = \tan(300^\circ - 5 \times 360^\circ) = \tan 300^\circ$

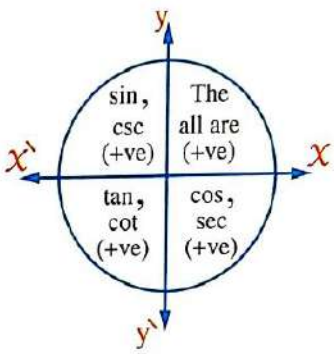
Signs of trigonometric functions

If $\angle AOB$ the directed is in its standard position and its terminal side intersects the unit circle at the point $B(x, y)$, and if $m(\angle AOB) = \theta$, then

$\angle AOB$ lies in one of the quadrants as follows :

First quadrant	Second quadrant	Third quadrant	Fourth quadrant
			
$\theta \in]0, \frac{\pi}{2}[$	$\theta \in]\frac{\pi}{2}, \pi[$	$\theta \in]\pi, \frac{3\pi}{2}[$	$\theta \in]\frac{3\pi}{2}, 2\pi[$
$x > 0, y > 0$	$x < 0, y > 0$	$x < 0, y < 0$	$x > 0, y < 0$
all the trigonometric functions are positive.	$\sin \theta$, $\csc \theta$ are positive and the other functions are negative.	$\tan \theta$, $\cot \theta$ are positive and the other functions are negative.	$\cos \theta$, $\sec \theta$ are positive and the other functions are negative.

- We can summarize the previous results in the figure and in the following table :

Quadrant	The interval that θ belongs to	sign of \cos, \sec	sign of \sin, \csc	sign of \tan, \cot	
First	$]0, \frac{\pi}{2}[$	+	+	+	
Second	$]\frac{\pi}{2}, \pi[$	-	+	-	
Third	$]\pi, \frac{3\pi}{2}[$	-	-	+	
Fourth	$]\frac{3\pi}{2}, 2\pi[$	+	-	-	

For example :

- $\tan 320^\circ$ is negative , because :

The angle of measure 320° lies in the fourth quadrant $270^\circ < 320^\circ < 360^\circ$

- $\sin 160^\circ$ is positive , because :

The angle of measure 160° lies in the second quadrant $90^\circ < 160^\circ < 180^\circ$

Remark

The trigonometric functions of the equivalent angles have the same sign.

Example 2

Determine the sign of each of the following trigonometric ratios :

1 $\sin 970^\circ$

2 $\cos \frac{7\pi}{3}$

3 $\tan (-200^\circ)$

4 $\csc \left(-\frac{8}{5}\pi\right)$

Solution

1 $\sin 970^\circ = \sin (250^\circ + 2 \times 360^\circ) = \sin 250^\circ$

, $\therefore 180^\circ < 250^\circ < 270^\circ$

i.e. This angle lies in the third quadrant.

$\therefore \sin 250^\circ$ is negative.

$\therefore \sin 970^\circ$ is negative.

$$2 \cos \frac{7}{3} \pi = \cos \left(\frac{7}{3} \times 180^\circ \right) = \cos 420^\circ = \cos (60^\circ + 360^\circ) = \cos 60^\circ$$

$$, \because 0^\circ < 60^\circ < 90^\circ$$

i.e. This angle lies in the first quadrant.

$\therefore \cos 60^\circ$ is positive.

$\therefore \cos \frac{7}{3} \pi$ is positive.

$$3 \tan (-200^\circ) = \tan (-200^\circ + 360^\circ) = \tan 160^\circ$$

$$, \because 90^\circ < 160^\circ < 180^\circ$$

i.e. This angle lies in the second quadrant.

$\therefore \tan 160^\circ$ is negative.

$\therefore \tan (-200^\circ)$ is negative.

$$4 \csc \left(-\frac{8}{5} \pi \right) = \csc \left(-\frac{8}{5} \times 180^\circ \right) = \csc (-288^\circ) = \csc (-288^\circ + 360^\circ) = \csc 72^\circ$$

$$, \because 0^\circ < 72^\circ < 90^\circ$$

i.e. This angle lies in the first quadrant.

$\therefore \csc 72^\circ$ is positive.

$\therefore \csc \left(-\frac{8}{5} \pi \right)$ is positive.

TRY TO SOLVE

Determine the sign of each of the following trigonometric ratios :

1 $\cos 620^\circ$

2 $\sec (-30^\circ)$

3 $\cot \frac{11}{3} \pi$

Example 3

If B $\left(x, \frac{1}{2} \right)$ is the point of intersection of the terminal side of the directed angle of measure θ in its standard position with the unit circle where $90^\circ < \theta < 180^\circ$, find the value of each of : $\cos \theta$ and $\tan \theta$

Solution

$$\because 90^\circ < \theta < 180^\circ$$

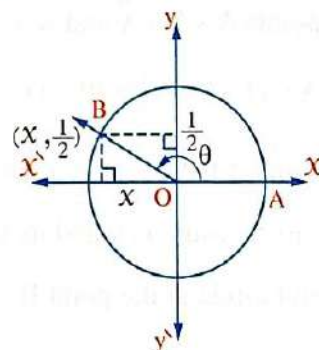
\therefore B lies in the second quadrant

$$, \because \text{for any point } (X, y) \text{ on the unit circle, we get } X^2 + y^2 = 1$$

$$\therefore X^2 + \left(\frac{1}{2} \right)^2 = 1$$

$$\therefore X^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore X = \pm \frac{\sqrt{3}}{2}$$



, \therefore The point B $(x, \frac{1}{2})$ lies in the second quadrant.

$$\therefore x = -\frac{\sqrt{3}}{2} \qquad \therefore B = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\therefore \cos \theta = \frac{-\sqrt{3}}{2}, \tan \theta = \frac{y}{x} = -\frac{1}{\sqrt{3}}$$

Example 4

If $\theta \in \left[\frac{3\pi}{2}, 2\pi\right]$, $\cos \theta = \frac{5}{13}$, then find all trigonometric functions of θ

Solution

Let $m(\angle AOB) = \theta$ where θ is in the 4th quadrant and the coordinates of B are (x, y)

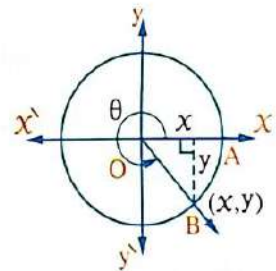
$$\therefore x = \cos \theta = \frac{5}{13}, y = \sin \theta \text{ where } \sin \theta < 0$$

$$\therefore x^2 + y^2 = 1 \qquad \therefore \left(\frac{5}{13}\right)^2 + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = 1 - \frac{25}{169} = \frac{144}{169} \qquad \therefore \sin \theta = -\frac{12}{13} \qquad \therefore B = \left(\frac{5}{13}, -\frac{12}{13}\right)$$

$$\text{, then we get : } \tan \theta = \frac{y}{x} = -\frac{12}{5},$$

$$\csc \theta = \frac{1}{y} = -\frac{13}{12}, \sec \theta = \frac{1}{x} = \frac{13}{5} \text{ and } \cot \theta = \frac{x}{y} = -\frac{5}{12}$$



TRY TO SOLVE

If $180^\circ < \theta < 270^\circ$, $\cos \theta = -\frac{4}{5}$, then find all trigonometric functions of θ

The trigonometric ratios of some special angles

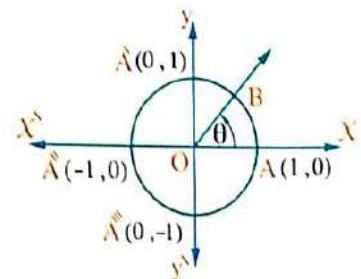
If the unit circle intersects the two coordinate axes at the points A, \hat{A} , \check{A} and \bar{A} as shown in the opposite figure,

$$A = (1, 0), \hat{A} = (0, 1), \check{A} = (-1, 0) \text{ and } \bar{A} = (0, -1)$$

, and if the directed angle is in its standard position

, its measure is θ and its terminal side intersects the

unit circle at the point B, then we notice that :



First If $\theta = 0^\circ$ or 360° (i.e. $\theta = 0$ or 2π)

Then the point B coincides with point A (1, 0), then we get :

$$\begin{aligned}\cos 0^\circ &= \cos 360^\circ = 1 & , \sec 0^\circ &= \sec 360^\circ = 1 \\ , \sin 0^\circ &= \sin 360^\circ = 0 & , \csc 0^\circ &= \csc 360^\circ = \frac{1}{0} \text{ (undefined)} \\ , \tan 0^\circ &= \tan 360^\circ = 0 & , \cot 0^\circ &= \cot 360^\circ = \frac{1}{0} \text{ (undefined)}\end{aligned}$$

Second If $\theta = 90^\circ$ (i.e. $\theta = \frac{\pi}{2}$)

Then the point B coincides with the point A (0, 1), then we get :

$$\begin{aligned}\cos 90^\circ &= 0 & , \sec 90^\circ &= \frac{1}{0} \text{ (undefined)} \\ , \sin 90^\circ &= 1 & , \csc 90^\circ &= 1 \\ , \tan 90^\circ &= \frac{1}{0} \text{ (undefined)} & , \cot 90^\circ &= 0\end{aligned}$$

Third If $\theta = 180^\circ$ (i.e. $\theta = \pi$)

Then the point B coincides with the point A (-1, 0), then we get :

$$\begin{aligned}\cos 180^\circ &= -1 & , \sec 180^\circ &= -1 \\ , \sin 180^\circ &= 0 & , \csc 180^\circ &= \frac{1}{0} \text{ (undefined)} \\ , \tan 180^\circ &= 0 & , \cot 180^\circ &= \frac{-1}{0} \text{ (undefined)}\end{aligned}$$

Fourth If $\theta = 270^\circ$ (i.e. $\theta = \frac{3\pi}{2}$)

Then the point B coincides with the point A (0, -1), then we get :

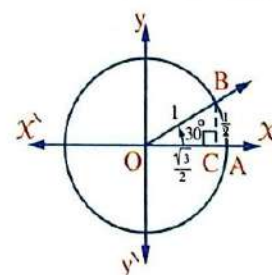
$$\begin{aligned}\cos 270^\circ &= 0 & , \sec 270^\circ &= \frac{1}{0} \text{ (undefined)} \\ , \sin 270^\circ &= -1 & , \csc 270^\circ &= -1 \\ , \tan 270^\circ &= \frac{-1}{0} \text{ (undefined)} & , \cot 270^\circ &= 0\end{aligned}$$

Fifth If $\theta = 30^\circ$ (i.e. $\theta = \frac{\pi}{6}$)

From the geometry of the figure notice that :

OB = 1 length unit (because B lies on the unit circle)

, then B $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so :



$$\begin{aligned}\cos 30^\circ &= \frac{\sqrt{3}}{2} & , \sec 30^\circ &= \frac{2}{\sqrt{3}} \\ \sin 30^\circ &= \frac{1}{2} & , \csc 30^\circ &= 2 \\ \tan 30^\circ &= \frac{1}{\sqrt{3}} & , \cot 30^\circ &= \sqrt{3}\end{aligned}$$

Sixth If $\theta = 60^\circ$ (i.e. $\theta = \frac{\pi}{3}$)

From the geometry of the figure notice that :

OB = 1 length unit (because B lies on the unit circle)

, then B $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so :

$$\begin{aligned}\cos 60^\circ &= \frac{1}{2} & , \sec 60^\circ &= 2 \\ \sin 60^\circ &= \frac{\sqrt{3}}{2} & , \csc 60^\circ &= \frac{2}{\sqrt{3}} \\ \tan 60^\circ &= \sqrt{3} & , \cot 60^\circ &= \frac{1}{\sqrt{3}}\end{aligned}$$

Seventh If $\theta = 45^\circ$ (i.e. $\theta = \frac{\pi}{4}$)

From the geometry of the figure notice that :

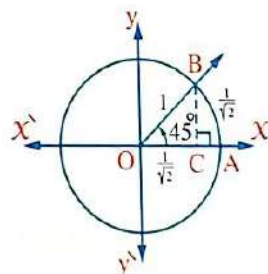
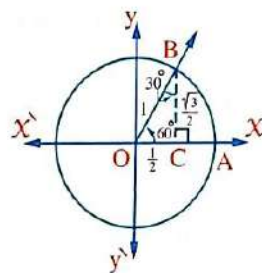
OB = 1 length unit (because B lies on the unit circle)

, then B $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, so :

$$\begin{aligned}\cos 45^\circ &= \frac{1}{\sqrt{2}} & , \sec 45^\circ &= \sqrt{2} \\ \sin 45^\circ &= \frac{1}{\sqrt{2}} & , \csc 45^\circ &= \sqrt{2} \\ \tan 45^\circ &= 1 & , \cot 45^\circ &= 1\end{aligned}$$

Notice that

Length of the opposite side to the angle whose measure is $30^\circ = \frac{1}{2}$ the length of the hypotenuse



Notice that

OC = CB
Because $\triangle OBC$ is an isosceles triangle.

- We can summarize the previous results in the following table :

the measure of θ	The point of the intersection of the terminal side with the unit circle	The values of the trigonometric functions		
		$\sin \theta$	$\cos \theta$	$\tan \theta$
0° or 360°	$(1, 0)$	0	1	0
90°	$(0, 1)$	1	0	undefined
180°	$(-1, 0)$	0	-1	0
270°	$(0, -1)$	-1	0	undefined
30°	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
60°	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
45°	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1

Example 5

Find the value of :

$$4 \sin 30^\circ \sin 90^\circ - \cos 0^\circ \sec 60^\circ + 5 \tan 45^\circ + 10 \cos^2 45^\circ \sin 270^\circ - \tan 30^\circ \sin 180^\circ$$

Solution

$$\begin{aligned} \text{The expression} &= 4 \times \frac{1}{2} \times 1 - 1 \times 2 + 5 \times 1 + 10 \times \left(\frac{1}{\sqrt{2}}\right)^2 \times (-1) - \frac{1}{\sqrt{3}} \times 0 \\ &= 2 - 2 + 5 - 5 - 0 = 0 \end{aligned}$$

Example 6

Prove that :

$$\sin^2 60^\circ + \sin^2 45^\circ + \sin^2 30^\circ = \cos^2 \frac{\pi}{6} \sin \frac{\pi}{2} - \frac{1}{3} \tan^2 \frac{\pi}{3} \cos \pi + \cos^2 \frac{\pi}{3} \sin \frac{3\pi}{2}$$

Solution

$$\text{The left hand side} = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{2}$$

$$\begin{aligned}\text{The right hand side} &= \cos^2 30^\circ \sin 90^\circ - \frac{1}{3} \tan^2 60^\circ \cos 180^\circ + \cos^2 60^\circ \sin 270^\circ \\ &= \left(\frac{\sqrt{3}}{2}\right)^2 \times 1 - \frac{1}{3} \times (\sqrt{3})^2 \times (-1) + \left(\frac{1}{2}\right)^2 \times (-1) = \frac{3}{4} + 1 - \frac{1}{4} = \frac{3}{2}\end{aligned}$$

\therefore The two sides are equal.

Example 7

Find the value of X which satisfies : $X \sin \frac{\pi}{6} \cos^2 \frac{\pi}{4} = \cos^2 30^\circ \sin \frac{\pi}{2}$

Solution

$$\begin{aligned}\therefore X \sin 30^\circ \cos^2 45^\circ &= \cos^2 30^\circ \sin 90^\circ & \therefore X \times \frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)^2 &= \left(\frac{\sqrt{3}}{2}\right)^2 \times 1 \\ \therefore \frac{1}{4} X &= \frac{3}{4} & \therefore X &= 3\end{aligned}$$

Example 8

If $X \in]0^\circ, 90^\circ[$, find the value of X that satisfies :

$$\sin X \sec^2 45^\circ = \tan^2 60^\circ - 2 \cos 360^\circ$$

Solution

$$\begin{aligned}\therefore \sin X \sec^2 45^\circ &= \tan^2 60^\circ - 2 \cos 360^\circ \\ \therefore \sin X \times (\sqrt{2})^2 &= (\sqrt{3})^2 - 2 \times 1 & \therefore 2 \times \sin X &= 3 - 2 = 1 \\ \therefore \sin X &= \frac{1}{2} & \therefore X &= 30^\circ\end{aligned}$$

TRY TO SOLVE

1 Find the value of :

$$\cos 90^\circ \csc 30^\circ + \sec^2 45^\circ \sin 30^\circ - \cos 270^\circ \sin 180^\circ$$

2 If $X \in [0^\circ, 90^\circ]$, find the value of X which satisfies :

$$\cos X = \sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ$$

Lesson

4

Related angles

Definition of the related angles

They are two angles the difference between their measures or the sum of their measures equals a whole number of right angles.

For example : The two angles of measures 30° , 210° are two related angles.

because : $210^\circ - 30^\circ = 180^\circ$ **i.e.** two right angles.

The relation between trigonometric functions of related angles

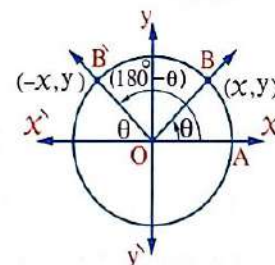
If the terminal side of the directed angle $\angle AOB$ in its standard position intersects the unit circle at the point $B(x, y)$ and $m(\angle AOB) = \theta$ such that $0^\circ < \theta < 90^\circ$, then :

1 Relation between trigonometric functions of related angles of measures θ , $(180^\circ - \theta)$:

If $\vec{B}(-x, y)$ is the image of the point $B(x, y)$ by reflection in the y-axis

, then $m(\angle AOB')$ the directed $= (180^\circ - \theta)$ thus :

$$\begin{aligned} \sin(180^\circ - \theta) &= \sin \theta & , & \quad \csc(180^\circ - \theta) = \csc \theta \\ \cos(180^\circ - \theta) &= -\cos \theta & , & \quad \sec(180^\circ - \theta) = -\sec \theta \\ \tan(180^\circ - \theta) &= -\tan \theta & , & \quad \cot(180^\circ - \theta) = -\cot \theta \end{aligned}$$



For example : • $\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$

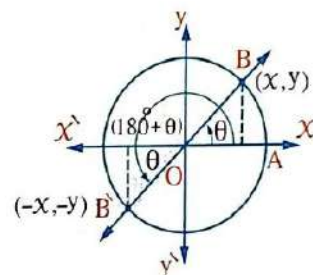
• $\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

• $\cot 135^\circ = \cot(180^\circ - 45^\circ) = -\cot 45^\circ = -1$

2 Relation between trigonometric functions of related angles of measures θ , $(180^\circ + \theta)$:

If $\vec{B}(-x, -y)$ is the image of the point $B(x, y)$ by reflection in the origin point, then $m(\angle AOB)$ the directed = $(180^\circ + \theta)$ thus:

$$\begin{aligned}\sin(180^\circ + \theta) &= -\sin \theta & \csc(180^\circ + \theta) &= -\csc \theta \\ \cos(180^\circ + \theta) &= -\cos \theta & \sec(180^\circ + \theta) &= -\sec \theta \\ \tan(180^\circ + \theta) &= \tan \theta & \cot(180^\circ + \theta) &= \cot \theta\end{aligned}$$



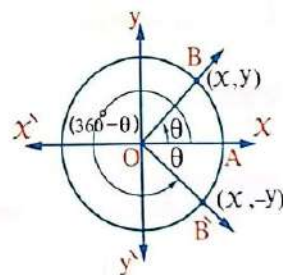
For example:

- $\sin 225^\circ = \sin(180^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$
- $\sec 210^\circ = \sec(180^\circ + 30^\circ) = -\sec 30^\circ = -\frac{2}{\sqrt{3}}$
- $\tan 240^\circ = \tan(180^\circ + 60^\circ) = \tan 60^\circ = \sqrt{3}$

3 Relation between trigonometric functions of related angles of measures θ , $(360^\circ - \theta)$:

If $\vec{B}(x, -y)$ is the image of the point $B(x, y)$ by reflection in the x -axis, then $m(\angle AOB)$ the directed = $(360^\circ - \theta)$ thus:

$$\begin{aligned}\sin(360^\circ - \theta) &= -\sin \theta & \csc(360^\circ - \theta) &= -\csc \theta \\ \cos(360^\circ - \theta) &= \cos \theta & \sec(360^\circ - \theta) &= \sec \theta \\ \tan(360^\circ - \theta) &= -\tan \theta & \cot(360^\circ - \theta) &= -\cot \theta\end{aligned}$$



For example:

- $\sin 300^\circ = \sin(360^\circ - 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$
- $\tan 315^\circ = \tan(360^\circ - 45^\circ) = -\tan 45^\circ = -1$
- $\sec 330^\circ = \sec(360^\circ - 30^\circ) = \sec 30^\circ = \frac{2}{\sqrt{3}}$

Note

The angle of measure $(-\theta)$ is equivalent to the angle of measure $(360^\circ - \theta)$

From this, we can deduce:

The relation between trigonometric functions of related angles of measures θ , $(-\theta)$ as follows:

$$\begin{aligned}\sin(-\theta) &= -\sin \theta & \csc(-\theta) &= -\csc \theta \\ \cos(-\theta) &= \cos \theta & \sec(-\theta) &= \sec \theta \\ \tan(-\theta) &= -\tan \theta & \cot(-\theta) &= -\cot \theta\end{aligned}$$

For example : • $\sin(-45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$

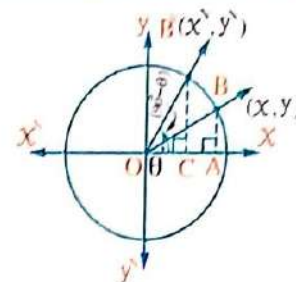
• $\cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}$

• $\cot(-30^\circ) = -\cot 30^\circ = -\sqrt{3}$

4 Relation between trigonometric functions of related angles of measures θ , $(90^\circ - \theta)$:

In the opposite figure :

The terminal side of the directed angle of measure $(90^\circ - \theta)$ in the standard position intersects the unit circle at the point $\hat{B}(\hat{x}, \hat{y})$



From the figure geometry, we find that :

$$\triangle C\hat{B}O \equiv \triangle AOB$$

$$\therefore C\hat{B} = AO \quad , \quad \text{then } \hat{y} = x$$

i.e. $\sin(90^\circ - \theta) = \cos \theta$

$$, CO = AB \quad , \quad \text{then } \hat{x} = y$$

i.e. $\cos(90^\circ - \theta) = \sin \theta$

$$, \therefore \tan(90^\circ - \theta) = \frac{\hat{y}}{\hat{x}} = \frac{x}{y}$$

$$\therefore \tan(90^\circ - \theta) = \cot \theta$$

Similarly, it is possible to deduce the relations between the reciprocals of the trigonometric functions of the two angles of measures θ , $(90^\circ - \theta)$ as follows :

$$\sin(90^\circ - \theta) = \cos \theta \quad , \quad \csc(90^\circ - \theta) = \sec \theta$$

$$\cos(90^\circ - \theta) = \sin \theta \quad , \quad \sec(90^\circ - \theta) = \csc \theta$$

$$\tan(90^\circ - \theta) = \cot \theta \quad , \quad \cot(90^\circ - \theta) = \tan \theta$$

For example : • $\sin 70^\circ = \sin(90^\circ - 20^\circ) = \cos 20^\circ$

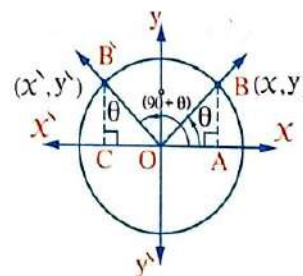
$$\bullet \frac{\sin 40^\circ}{\cos 50^\circ} = \frac{\sin(90^\circ - 50^\circ)}{\cos 50^\circ} = \frac{\cos 50^\circ}{\cos 50^\circ} = 1$$

$$\bullet \tan 10^\circ - \cot 80^\circ = \tan(90^\circ - 80^\circ) - \cot 80^\circ = \cot 80^\circ - \cot 80^\circ = 0$$

5 Relation between trigonometric functions of related angles of measures θ , $(90^\circ + \theta)$:

In the opposite figure :

The terminal side of the directed angle of measure $(90^\circ + \theta)$ in the standard position intersects the unit circle at the point $\hat{B}(\hat{x}, \hat{y})$



From the figure geometry , we find that :

$$\triangle COB \equiv \triangle ABO$$

$$\therefore CB = AO \quad , \quad \text{then } \hat{y} = x$$

$$, OC = AB \quad , \quad \text{then } \hat{x} = -y$$

$$, \therefore \tan (90^\circ + \theta) = \frac{\hat{y}}{\hat{x}} = \frac{x}{-y}$$

$$\text{i.e. } \sin (90^\circ + \theta) = \cos \theta$$

$$\text{i.e. } \cos (90^\circ + \theta) = -\sin \theta$$

$$\therefore \tan (90^\circ + \theta) = -\cot \theta$$

Similarly , it is possible to deduce the relations between the reciprocals of the trigonometric functions of the two angles of measures θ , $(90^\circ + \theta)$ as follows :

$$\sin (90^\circ + \theta) = \cos \theta \quad , \quad \csc (90^\circ + \theta) = \sec \theta$$

$$\cos (90^\circ + \theta) = -\sin \theta \quad , \quad \sec (90^\circ + \theta) = -\csc \theta$$

$$\tan (90^\circ + \theta) = -\cot \theta \quad , \quad \cot (90^\circ + \theta) = -\tan \theta$$

For example : • $\sin 120^\circ = \sin (90^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

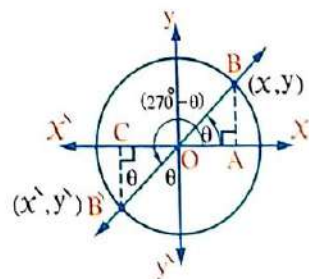
• $\cos 150^\circ = \cos (90^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

• $\cot 135^\circ = \cot (90^\circ + 45^\circ) = -\tan 45^\circ = -1$

6 Relation between trigonometric functions of related angles of measures θ , $(270^\circ - \theta)$:

In the opposite figure :

The terminal side of the directed angle of measure $(270^\circ - \theta)$ in the standard position intersects the unit circle at the point $B(\hat{x}, \hat{y})$



From the figure geometry , we find that :

$$\triangle COB \equiv \triangle ABO$$

$$\therefore CB = AO \quad , \quad \text{then } \hat{y} = -x$$

$$, CO = AB \quad , \quad \text{then } \hat{x} = -y$$

$$, \therefore \tan (270^\circ - \theta) = \frac{\hat{y}}{\hat{x}} = \frac{-x}{-y} = \frac{x}{y}$$

$$\text{i.e. } \sin (270^\circ - \theta) = -\cos \theta$$

$$\text{i.e. } \cos (270^\circ - \theta) = -\sin \theta$$

$$\therefore \tan (270^\circ - \theta) = \cot \theta$$

Similarly, it is possible to deduce the relations between the reciprocals of the trigonometric functions of the two angles of measures θ , $(270^\circ - \theta)$ as follows :

$$\begin{aligned} \sin(270^\circ - \theta) &= -\cos \theta & , & & \csc(270^\circ - \theta) &= -\sec \theta \\ \cos(270^\circ - \theta) &= -\sin \theta & , & & \sec(270^\circ - \theta) &= -\csc \theta \\ \tan(270^\circ - \theta) &= \cot \theta & , & & \cot(270^\circ - \theta) &= \tan \theta \end{aligned}$$

For example : • $\sin 225^\circ = \sin(270^\circ - 45^\circ) = -\cos 45^\circ = \frac{-1}{\sqrt{2}}$

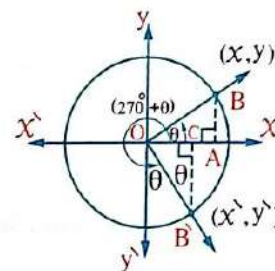
• $\tan 240^\circ = \tan(270^\circ - 30^\circ) = \cot 30^\circ = \sqrt{3}$

• $\csc 210^\circ = \csc(270^\circ - 60^\circ) = -\sec 60^\circ = -2$

7 Relation between trigonometric functions of related angles of measures θ , $(270^\circ + \theta)$:

In the opposite figure :

The terminal side of the directed angle of measure $(270^\circ + \theta)$ in the standard position intersects the unit circle at the point $\hat{B}(\hat{x}, \hat{y})$



From the figure geometry, we find that :

$$\Delta COB \equiv \Delta ABO$$

$$\therefore CB = AO \quad , \quad \text{then } \hat{y} = -x$$

$$, CO = AB \quad , \quad \text{then } \hat{x} = y$$

$$, \therefore \tan(270^\circ + \theta) = \frac{\hat{y}}{\hat{x}} = \frac{-x}{y}$$

i.e. $\sin(270^\circ + \theta) = -\cos \theta$

i.e. $\cos(270^\circ + \theta) = \sin \theta$

$$\therefore \tan(270^\circ + \theta) = -\cot \theta$$

Similarly, it is possible to deduce the relations between the reciprocals of the trigonometric functions of the two angles of measures θ , $(270^\circ + \theta)$ as follows :

$$\sin(270^\circ + \theta) = -\cos \theta \quad , \quad \csc(270^\circ + \theta) = -\sec \theta$$

$$\cos(270^\circ + \theta) = \sin \theta \quad , \quad \sec(270^\circ + \theta) = \csc \theta$$

$$\tan(270^\circ + \theta) = -\cot \theta \quad , \quad \cot(270^\circ + \theta) = -\tan \theta$$

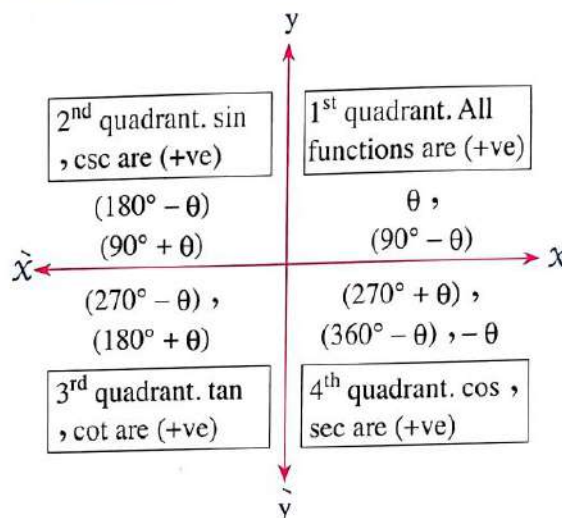
For example : • $\sin 300^\circ = \sin(270^\circ + 30^\circ) = -\cos 30^\circ = \frac{-\sqrt{3}}{2}$

• $\sec 330^\circ = \sec(270^\circ + 60^\circ) = \csc 60^\circ = \frac{2}{\sqrt{3}}$

• $\cot 315^\circ = \cot(270^\circ + 45^\circ) = -\tan 45^\circ = -1$

We can summarize all the previous relations as follows (where θ is the measure of an acute angle):

- 1 We can determine the sign of the trigonometric function according to the quadrant in which the terminal side of the angle in its standard position lies as shown in the opposite figure.
- 2 The numerical values of the same trigonometric function of the angles whose measures are θ , $(180^\circ - \theta)$, $(180^\circ + \theta)$, $(360^\circ - \theta)$, $(-\theta)$ are equal and differs only in the sign according to the quadrant in which the terminal side of the angle in its standard position lies.



For example :

$\sin (360^\circ - \theta)$	=	-	\sin	θ
\Downarrow		\Downarrow	\Downarrow	\Downarrow
the angle lies in the 4 th quadrant where sin is (-ve)		put the (-ve) sign	the function does not change	change the angle to θ

- 3 For the trigonometric functions of angles whose measures are : $(90^\circ - \theta)$, $(90^\circ + \theta)$, $(270^\circ - \theta)$ and $(270^\circ + \theta)$

Change the trigonometric function as follows : $\sin \Leftrightarrow \cos$, $\cot \Leftrightarrow \tan$, $\sec \Leftrightarrow \csc$, taking into account the sign according to the quadrant that the angle lies before changing the trigonometric function.

For example :

$\tan (270^\circ - \theta)$	=	+	\cot	θ
\Downarrow		\Downarrow	\Downarrow	\Downarrow
the angle lies in the 3 rd quad. where tan is (+ve)		put the (+ve) sign	change the function from "tan" to "cot"	change the angle to θ

Finding a trigonometric function of an angle whose measure is given (α)

First If $0^\circ < \alpha < 360^\circ$ i.e. $\alpha \in]0, 2\pi[$

- 1 We determine the quadrant in which the angle lies, then determine the sign of the trigonometric function.
- 2 We convert the trigonometric function of α into the same trigonometric function of the angle θ and $\theta \in]0, \frac{\pi}{2}[$ as follows :
 - Put α in the form $(180^\circ - \theta)$ if α lies in the 2nd quadrant.
 - Put α in the form $(180^\circ + \theta)$ if α lies in the 3rd quadrant.
 - Put α in the form $(360^\circ - \theta)$ if α lies in the 4th quadrant.

Second If $\alpha > 360^\circ$ i.e. $\alpha > 2\pi$

- 1 Put α in the form of $(2n\pi + \theta)$ where $\theta \in]0, 2\pi[$, n is a positive integer, then the trigonometric function of the angle α is the same of the angle θ
- 2 Find the trigonometric function of the angle θ as in the first.

Third If α is (-ve) i.e. $\alpha < 0^\circ$

We follow one of the following two methods :

The first method

Apply the rule of the trigonometric function of the angle whose measure is negative, that is : $\sin(-\theta) = -\sin \theta$, $\cos(-\theta) = \cos \theta$, $\tan(-\theta) = -\tan \theta$ and so on, then we find the trigonometric function of the angle θ as in the first and the second.

The second method

Add to α an integer number of 2π (i.e. add to α the measures $360^\circ n$ or $2\pi n$ where $n \in \mathbb{Z}^+$) to get a positive angle $\theta \in]0, 2\pi[$, then we get the trigonometric function of the angle θ , the result is the same trigonometric function of the negative angle α

Example 1

Find the value of each of the following :

1 $\sin 240^\circ$

2 $\cos \frac{5\pi}{3}$

3 $\cos 570^\circ$

4 $\tan (-150^\circ)$

Solution

1 $\sin 240^\circ = \sin (180^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$

2 $\cos \frac{5\pi}{3} = \cos \left(\frac{5 \times 180^\circ}{3} \right) = \cos 300^\circ = \cos (360^\circ - 60^\circ) = \cos 60^\circ = \frac{1}{2}$

or $\cos \frac{5\pi}{3} = \cos \left(2\pi - \frac{\pi}{3} \right) = \cos \frac{\pi}{3} = \frac{1}{2}$

3 $\cos 570^\circ = \cos (360^\circ + 210^\circ) = \cos 210^\circ = \cos (180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

4 $\tan (-150^\circ) = -\tan 150^\circ = -\tan (180^\circ - 30^\circ) = -(-\tan 30^\circ) = \tan 30^\circ = \frac{1}{\sqrt{3}}$

Example 2

Find the value of each of the following in two different methods :

1 $\sin 120^\circ$

2 $\cot 135^\circ$

3 $\cos (-240^\circ)$

4 $\sec \frac{15\pi}{4}$

Solution

1 $\sin 120^\circ = \sin (180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

or $\sin 120^\circ = \sin (90^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

2 $\cot 135^\circ = \cot (180^\circ - 45^\circ) = -\cot 45^\circ = -1$

or $\cot 135^\circ = \cot (90^\circ + 45^\circ) = -\tan 45^\circ = -1$

3 $\cos (-240^\circ) = \cos 240^\circ = \cos (180^\circ + 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

or $\cos (-240^\circ) = \cos 240^\circ = \cos (270^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

4 $\sec \frac{15\pi}{4} = \sec \left(\frac{15 \times 180^\circ}{4} \right) = \sec 675^\circ = \sec (360^\circ + 315^\circ) = \sec 315^\circ$

$$= \sec (360^\circ - 45^\circ) = \sec 45^\circ = \sqrt{2}$$

or $\sec \frac{15\pi}{4} = \sec 315^\circ = \sec (270^\circ + 45^\circ) = \csc 45^\circ = \sqrt{2}$

Example 3

Without using the calculator, find the value of the following :

$$\cos(-150^\circ) \sin 600^\circ + \cos \frac{2\pi}{3} \sin 330^\circ - \sec\left(\frac{-5\pi}{4}\right) \tan 900^\circ$$

Solution

$$\because \cos(-150^\circ) = \cos 150^\circ = \cos(180^\circ - 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$, \sin 600^\circ = \sin(360^\circ + 240^\circ) = \sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

$$, \cos \frac{2\pi}{3} = \cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$$

$$, \sin 330^\circ = \sin(360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

$$, \sec\left(\frac{-5\pi}{4}\right) = \sec \frac{5\pi}{4} = \sec 225^\circ = \sec(180^\circ + 45^\circ) = -\sec 45^\circ = -\sqrt{2}$$

$$, \tan 900^\circ = \tan(720^\circ + 180^\circ) = \tan 180^\circ = \text{zero}$$

$$\begin{aligned} \therefore \text{The expression} &= \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right) - (-\sqrt{2})(\text{zero}) \\ &= \frac{3}{4} + \frac{1}{4} + \text{zero} = 1 \end{aligned}$$

TRY TO SOLVE

Without using the calculator :

1 Find the value of : $\cos 210^\circ \sin 510^\circ - \sin 330^\circ \cos(-330^\circ)$

2 Prove that : $\sin 600^\circ \cos(-390^\circ) + \sin 150^\circ \cos(-240^\circ) = -1$

Example 4

If the directed angle of measure θ is in the standard position, and its terminal side passes through the point $\left(\frac{5}{13}, \frac{12}{13}\right)$, find the following trigonometric functions :

1 $\sin(90^\circ - \theta)$

2 $\cos(180^\circ + \theta)$

3 $\sec(90^\circ + \theta)$

4 $\csc(270^\circ - \theta)$

5 $\tan(360^\circ - \theta)$

6 $\cot(-\theta)$

Solution

$$\therefore x^2 + y^2 = \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 = \frac{25}{169} + \frac{144}{169} = 1$$

\therefore The point $\left(\frac{5}{13}, \frac{12}{13}\right) \in$ unit circle

1 $\sin(90^\circ - \theta) = \cos \theta = \frac{5}{13}$

3 $\sec(90^\circ + \theta) = -\csc \theta = -\frac{13}{12}$

5 $\tan(360^\circ - \theta) = -\tan \theta = -\frac{12}{5}$

2 $\cos(180^\circ + \theta) = -\cos \theta = -\frac{5}{13}$

4 $\csc(270^\circ - \theta) = -\sec \theta = -\frac{13}{5}$

6 $\cot(-\theta) = -\cot \theta = -\frac{5}{12}$

Example 5

If θ is the measure of an acute positive angle in its standard position and determines the point $B\left(\frac{3}{5}, y\right)$ on the unit circle, find :

1 $\tan(90^\circ - \theta) + \sec(90^\circ - \theta)$

2 $\cot(270^\circ + \theta) - \tan(90^\circ + \theta) - \sin(180^\circ + \theta)$

Solution

$$\therefore x^2 + y^2 = 1 \text{ for any point on the unit circle.}$$

$$\therefore \frac{9}{25} + y^2 = 1$$

$$\therefore y^2 = \frac{16}{25}$$

$$\therefore y = \frac{4}{5}, \text{ where } y > 0$$

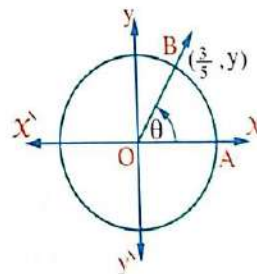
$$\therefore B = \left(\frac{3}{5}, \frac{4}{5}\right)$$

1 $\tan(90^\circ - \theta) + \sec(90^\circ - \theta) = \cot \theta + \csc \theta = \frac{3}{4} + \frac{5}{4} = \frac{8}{4} = 2$

2 $\cot(270^\circ + \theta) - \tan(90^\circ + \theta) - \sin(180^\circ + \theta)$

$$= -\tan \theta - (-\cot \theta) - (-\sin \theta)$$

$$= -\tan \theta + \cot \theta + \sin \theta = -\frac{4}{3} + \frac{3}{4} + \frac{4}{5} = \frac{13}{60}$$



Example 6

If $\cos \theta = \frac{-4}{5}$ where $\theta \in]90^\circ, 180^\circ[$, find the value of each of the following :

1 $\sin(180^\circ - \theta)$

2 $\sec(360^\circ - \theta)$

3 $\cos(-\theta)$

4 $\tan(\theta - 180^\circ)$

Solution

Let $m(\angle AOB) = \theta$, where $\theta \in]90^\circ, 180^\circ[$

as shown in the opposite figure and $B(x, y)$

$$\therefore x = \cos \theta = -\frac{4}{5}, y = \sin \theta, \text{ where } y > 0$$

$$\therefore x^2 + y^2 = 1$$

$$\therefore \left(-\frac{4}{5}\right)^2 + y^2 = 1$$

$$\therefore y^2 = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\therefore y = \frac{3}{5}$$

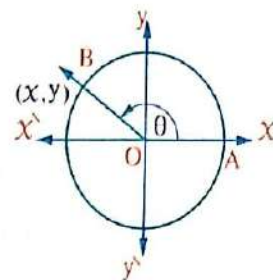
$$\therefore B = \left(-\frac{4}{5}, \frac{3}{5}\right)$$

$$1 \quad \sin(180^\circ - \theta) = \sin \theta = \frac{3}{5}$$

$$2 \quad \sec(360^\circ - \theta) = \sec \theta = -\frac{5}{4}$$

$$3 \quad \cos(-\theta) = \cos \theta = -\frac{4}{5}$$

$$4 \quad \tan(\theta - 180^\circ) = \tan(\theta - 180^\circ + 360^\circ) = \tan(180^\circ + \theta) = \tan \theta = -\frac{3}{4}$$



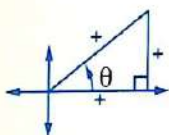
TRY TO SOLVE

If the terminal side of the directed angle of measure θ in its standard position intersects the unit circle at the point $(x, \frac{12}{13})$ such that $90^\circ < \theta < 180^\circ$, find the value of :

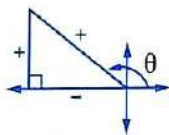
$$13 \cos(360^\circ - \theta) + \tan 225^\circ + \sec^2 300^\circ + 12 \tan(270^\circ - \theta)$$

Note

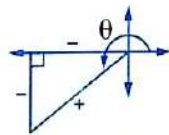
We can find the values of the trigonometric functions of an angle directly if we draw the angle in its standard position and we draw the right-angled triangle that represents it by using the value of the given trigonometric function concerning the signs according to the quadrant in which the angle lies as follows :



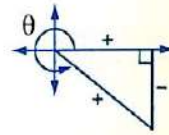
In the 1st
quadrant



In the 2nd
quadrant



In the 3rd
quadrant



In the 4th
quadrant

Example 7

If $\cos \alpha = \frac{7}{25}$ where α is the smallest positive angle, $\tan \beta = \frac{3}{4}$

, where β is the greatest positive angle where $0^\circ \leq \beta \leq 360^\circ$

Find the value of : $\cos(180^\circ + \alpha) \sin(\beta - 90^\circ) + \sin(360^\circ - \alpha) \sin(180^\circ - \beta)$

Solution

$$\therefore \cos \alpha < 0$$

$\therefore \alpha$ lies in the 2nd or 3rd quadrant

$\therefore \alpha$ is the smallest positive angle

$\therefore \alpha$ lies in the 2nd quadrant

$$\therefore \cos \alpha = \frac{-7}{25}$$

$$\therefore (MN)^2 = (25)^2 - (7)^2 = 576$$

$\therefore MN = 24$ length unit

$$\therefore \tan \beta > 0$$

$\therefore \beta$ lies in the 1st or 3rd quadrant.

$\therefore \beta$ is the greatest positive angle.

$\therefore \beta$ lies in the 3rd quadrant.

$$\therefore \tan \beta = \frac{3}{4}$$

$$\therefore (OQ)^2 = (3)^2 + (4)^2 = 25$$

$\therefore OQ = 5$ length unit

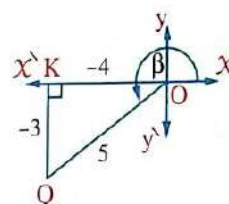
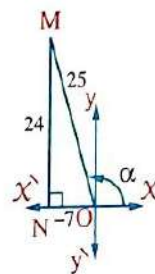
$$\therefore \text{The expression} = \cos (180^\circ + \alpha) \sin (\beta - 90^\circ) + \sin (360^\circ - \alpha) \sin (180^\circ - \beta)$$

$$= -\cos \alpha \sin (270^\circ + \beta) + (-\sin \alpha) \sin \beta$$

$$= (-\cos \alpha) (-\cos \beta) - \sin \alpha \sin \beta$$

$$= \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \frac{-7}{25} \times \left(\frac{-4}{5}\right) - \frac{24}{25} \times \frac{-3}{5} = \frac{28}{125} + \frac{72}{125} = \frac{100}{125} = \frac{4}{5}$$



Remark

If $\sin \alpha = \cos \beta$ or $\tan \alpha = \cot \beta$ or $\csc \alpha = \sec \beta$

, then $\alpha + \beta = 90^\circ$ such that α, β are the two measures of two acute positive angles.

For example : If $\tan 23^\circ = \cot \alpha$, then $23^\circ + \alpha = 90^\circ$ i.e. $\alpha = 67^\circ$

Example 8

If $\sin (3\theta + 28^\circ) = \cos (2\theta - 13^\circ)$, find one value of θ where $0^\circ < \theta < 90^\circ$

Solution

$$\therefore \sin (3\theta + 28^\circ) = \cos (2\theta - 13^\circ)$$

$$\therefore 3\theta + 28^\circ + 2\theta - 13^\circ = 90^\circ$$

$$\therefore 5\theta + 15^\circ = 90^\circ$$

$$\therefore 5\theta = 75^\circ$$

$$\therefore \theta = 15^\circ$$

Notice that

There are other values for θ such as $\theta = 49^\circ$ or $\theta = 87^\circ$ that satisfy the equation and to find these values we have to generalize the previous remark to get a general solution for this kind of equations.

Generalizing the previous remark

1 If $\sin \alpha = \cos \beta$, then $\sin \alpha = \sin (90^\circ - \beta)$

$$\therefore \alpha = 90^\circ - \beta \quad \text{or} \quad \alpha + 90^\circ - \beta = 180^\circ$$

$$\therefore \alpha + \beta = 90^\circ \quad \quad \quad \therefore \alpha - \beta = 90^\circ$$

we can add the multiplies of (360°) to the angle 90°

An Important Alert

On solving , we must start by sine angle α

2 In the same way , we can deduce the same rules if $\csc \alpha = \sec \beta$

3 If $\tan \alpha = \cot \beta$, then :

$$\tan \alpha = \tan (90^\circ - \beta) \quad \text{or} \quad \tan \alpha = \tan (270^\circ - \beta)$$

$$\therefore \alpha = 90^\circ - \beta \quad \quad \quad \therefore \alpha = 270^\circ - \beta$$

$$\therefore \alpha + \beta = 90^\circ \quad \quad \quad \therefore \alpha + \beta = 270^\circ$$

We can add the multiplies of (360°) to the angles 90° and 270°

So , the general solution for any two angles α , β could be written as follows :

The general solution to solve the equations in the form :
 $\sin \alpha = \cos \beta$ or $\csc \alpha = \sec \beta$ or $\tan \alpha = \cot \beta$

1 If $\sin \alpha = \cos \beta$

, then $\alpha \pm \beta = 90^\circ + 360^\circ n$

i.e. $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

i.e. the measure of angle of sine \pm the measure of angle of cosine $= 90^\circ + 360^\circ n$

2 If $\csc \alpha = \sec \beta$

, then $\alpha \pm \beta = 90^\circ + 360^\circ n$

i.e. $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

, $\alpha \neq n\pi$, $\beta \neq (2n+1)\frac{\pi}{2}$

3 If $\tan \alpha = \cot \beta$

, then $\alpha + \beta = 90^\circ + 180^\circ n$

i.e. $\alpha + \beta = \frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$

, $\alpha \neq (2n+1)\frac{\pi}{2}$, $\beta \neq n\pi$

Example 9

Find the general solution of the equation :

$\cos 2\theta = \sin 4\theta$, then find the values of θ where $\theta \in]0, \frac{\pi}{2}[$

Solution

$\therefore \cos 2\theta = \sin 4\theta$

$\therefore \sin 4\theta = \cos 2\theta$

$\therefore \alpha = 4\theta$, $\beta = 2\theta$

$\therefore 4\theta \pm 2\theta = \frac{\pi}{2} + 2\pi n$

\therefore either $6\theta = \frac{\pi}{2} + 2\pi n$

$\therefore \theta = \frac{\pi}{12} + \frac{\pi}{3}n$

or $2\theta = \frac{\pi}{2} + 2\pi n$

$\therefore \theta = \frac{\pi}{4} + \pi n$

\therefore The general solution is $\frac{\pi}{12} + \frac{\pi}{3}n$ or $\frac{\pi}{4} + \pi n$ where $n \in \mathbb{Z}$

at $n = 0$: $\therefore \theta = \frac{\pi}{12} \in]0, \frac{\pi}{2}[$ or $\theta = \frac{\pi}{4} \in]0, \frac{\pi}{2}[$

at $n = 1$: $\therefore \theta = \frac{\pi}{12} + \frac{\pi}{3} = \frac{5}{12}\pi \in]0, \frac{\pi}{2}[$ or $\theta = \frac{\pi}{4} + \pi = \frac{5}{4}\pi \notin]0, \frac{\pi}{2}[$

at $n = 2$: $\therefore \theta = \frac{\pi}{12} + \frac{2\pi}{3} = \frac{3}{4}\pi \notin]0, \frac{\pi}{2}[$

\therefore The values of θ are $\frac{\pi}{12}$, $\frac{\pi}{4}$, $\frac{5\pi}{12}$ **i.e.** 15° , 45° , 75°

TRY TO SOLVE

Find the general solution of the equation : $\sin 3\theta = \cos \theta$, then find all the values of θ where $\theta \in]0, \frac{\pi}{2}[$ which satisfy the equation.

Example 10

Find the solution set of each of the following equations :

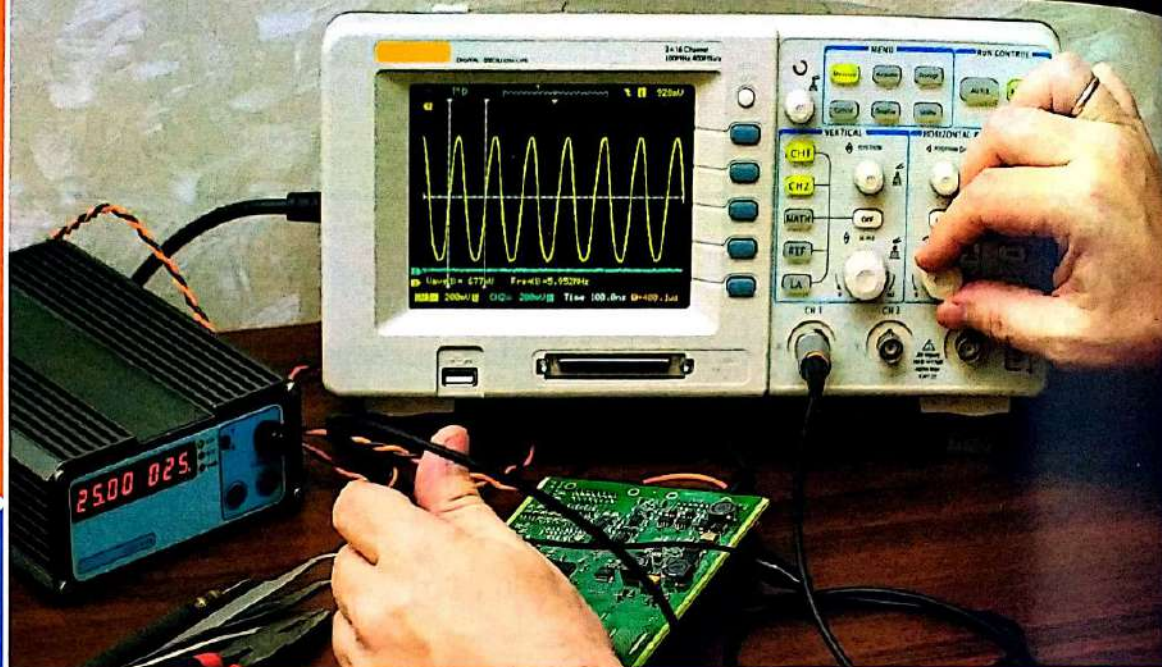
- 1 $2 \sin \theta - 1 = 0$ where $\theta \in]0, \frac{\pi}{2}[$
 2 $2 \cos \left(\frac{\pi}{2} - \theta \right) + \sqrt{3} = 0$ where $\theta \in]0, 2\pi[$
 3 $4 \cos^2 \theta - 3 = 0$ where $\theta \in]0, 2\pi[$

Solution

- 1 $\because 2 \sin \theta - 1 = 0$ $\therefore \sin \theta = \frac{1}{2}$ (positive)
 $\therefore \theta$ lies in the 1st or 2nd quadrant. \therefore The acute angle whose sine = $\frac{1}{2}$ is 30°
 $\therefore \theta = 30^\circ$ or $\theta = 180^\circ - 30^\circ = 150^\circ$ (refused because $\theta \in]0, \frac{\pi}{2}[$)
 \therefore The S.S = $\{30^\circ\}$
- 2 $\because 2 \cos \left(\frac{\pi}{2} - \theta \right) + \sqrt{3} = 0$ $\therefore 2 \sin \theta = -\sqrt{3}$
 $\therefore \sin \theta = \frac{-\sqrt{3}}{2}$ (negative) $\therefore \theta$ lies in the 3rd or 4th quadrant.
 \therefore the acute angle whose sine = $\frac{\sqrt{3}}{2}$ is 60°
 $\therefore \theta = 180^\circ + 60^\circ = 240^\circ$ or $\theta = 360^\circ - 60^\circ = 300^\circ$
 \therefore The S.S = $\{240^\circ, 300^\circ\}$
- 3 $\because 4 \cos^2 \theta - 3 = 0$ $\therefore 4 \cos^2 \theta = 3$
 $\therefore \cos^2 \theta = \frac{3}{4}$ $\therefore \cos \theta = \pm \frac{\sqrt{3}}{2}$
 \therefore Either $\cos \theta = \frac{\sqrt{3}}{2}$ (positive) $\therefore \theta$ lies in the 1st or 4th quadrant.
 \therefore the acute angle whose cosine = $\frac{\sqrt{3}}{2}$ is 30°
 $\therefore \theta = 30^\circ$ or $\theta = 360^\circ - 30^\circ = 330^\circ$
 or $\cos \theta = \frac{-\sqrt{3}}{2}$ (negative) $\therefore \theta$ lies in the 2nd or 3rd quadrant.
 $\therefore \theta = 180^\circ - 30^\circ = 150^\circ$ or $\theta = 180^\circ + 30^\circ = 210^\circ$
 \therefore The S.S = $\{30^\circ, 150^\circ, 210^\circ, 330^\circ\}$

Lesson

5



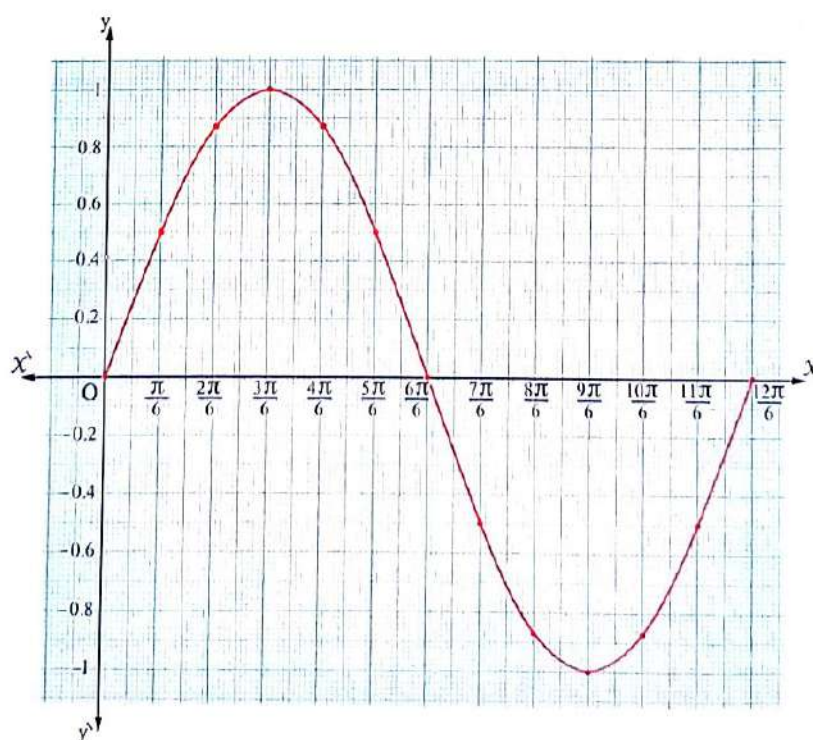
Graphing trigonometric functions

First sine function : $f : f(\theta) = \sin \theta$

To represent the function $f : f(\theta) = \sin \theta$ graphically ,
we form the following table for some special values of θ , where $\theta \in [0, 2\pi]$ and the
corresponding values of $\sin \theta$

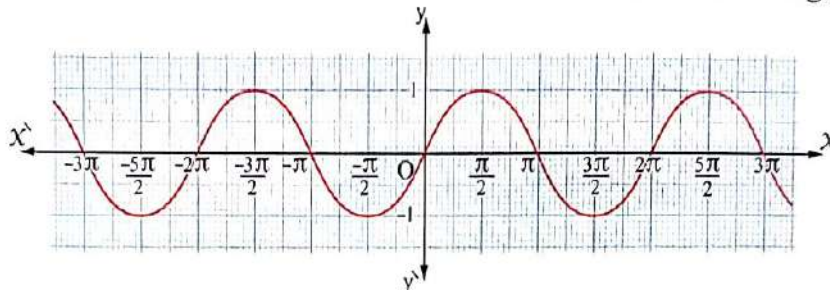
θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
$\sin \theta$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

Represent all of the points that we get in the table on the coordinate axes and join them
to get the curve of the function f on the interval $[0, 2\pi]$



We notice that : The function is periodic and its period is 2π (360°) where the curve of this function repeats itself on the intervals $[0, 2\pi]$, $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, ... and also on the intervals $[-2\pi, 0]$, $[-4\pi, -2\pi]$, $[-6\pi, -4\pi]$, ...

The general form of the curve of the sine function is as shown in the following graph :



From the previous , we can deduce the properties of the sine function $f : f(\theta) = \sin \theta$:

- 1 The domain of the sine function is $]-\infty, \infty[$
- 2 • The maximum value of the function is 1 and it happens when $\theta = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$
 • The minimum value of the function is -1 and it happens when $\theta = \frac{3\pi}{2} + 2n\pi, n \in \mathbb{Z}$
- 3 The range of the function = $[-1, 1]$
- 4 The function is periodic and its period is 2π (360°)

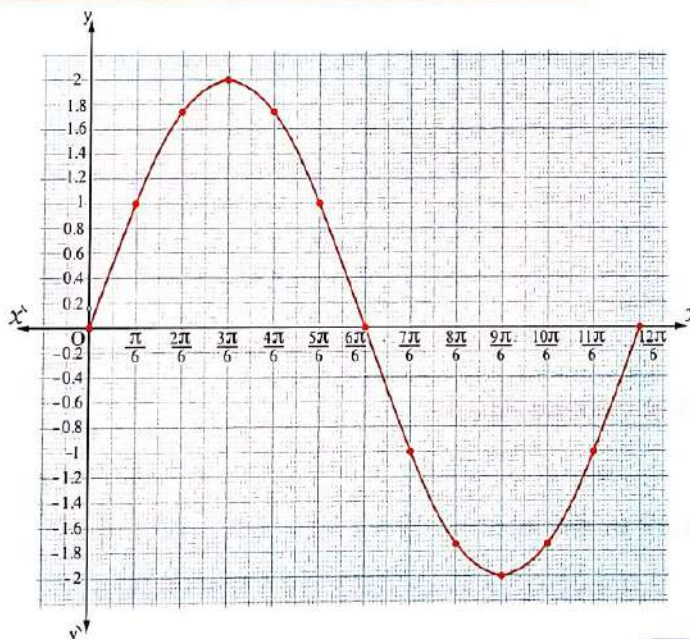
Example 1

Graph the function where $y = 2 \sin \theta$, where $\theta \in [0, 2\pi]$, then from the graph find the maximum and minimum values of the function, its range and its period.

Solution

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
y	0	1	1.7	2	1.7	1	0	-1	-1.7	-2	-1.7	-1	0

- The maximum value of the function = 2 ,
the minimum value of the function = -2
- The range of the function = $[-2, 2]$
- The period of the function = 2π (360°)



TRY TO SOLVE

Represent graphically the function $f : f(\theta) = 3 \sin \theta$, where $\theta \in [0, 2\pi]$, then from the graph find :

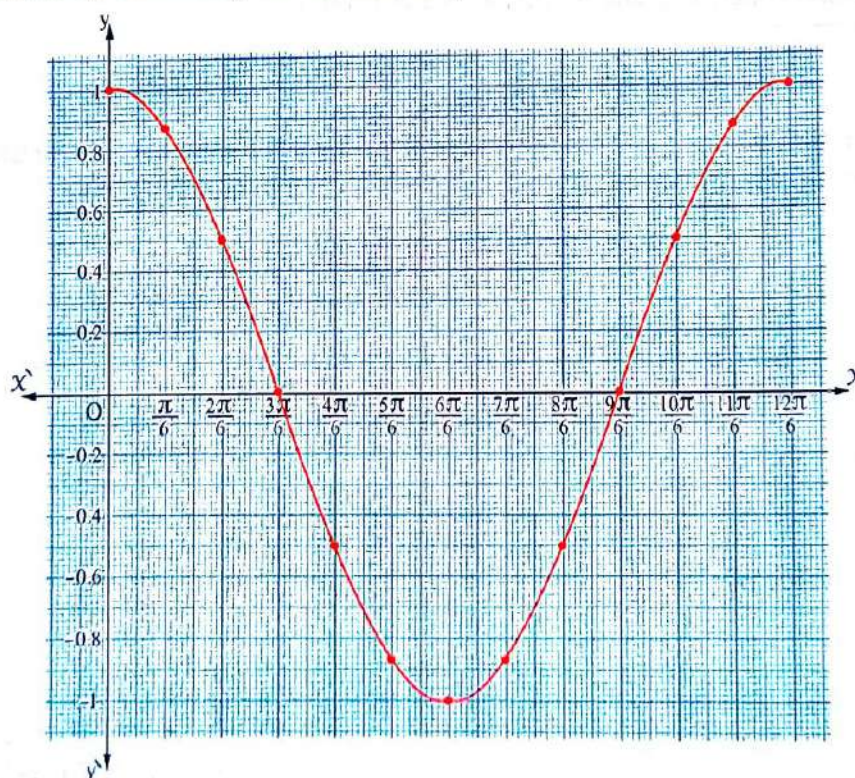
- 1 The maximum and minimum values of the function.
- 2 The range of the function.
- 3 The period of the function.

Second cosine function : $f : f(\theta) = \cos \theta$

To represent the function $f : f(\theta) = \cos \theta$ graphically, we form the following table for some special values of θ on the interval $[0, 2\pi]$ and the corresponding values of $\cos \theta$

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
$\cos \theta$	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1

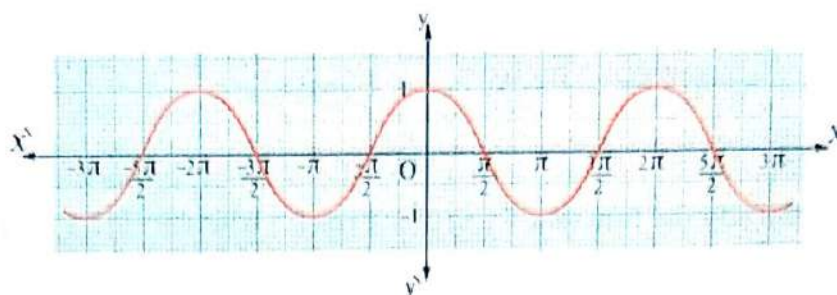
Represent all of the points that we get in the table on the coordinate axis and join them to get the curve of the function f on the interval $[0, 2\pi]$



We notice that :

The function is periodic and its period is 2π (360°) where the curve of this function repeats itself on the intervals $[0, 2\pi]$, $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, ... and also on the intervals $[-2\pi, 0]$, $[-4\pi, -2\pi]$, $[-6\pi, -4\pi]$, ...

The general form of the curve of the cosine function is as shown in the following graph :



From the previous , we can deduce the properties of the cosine function $f : f(\theta) = \cos \theta$:

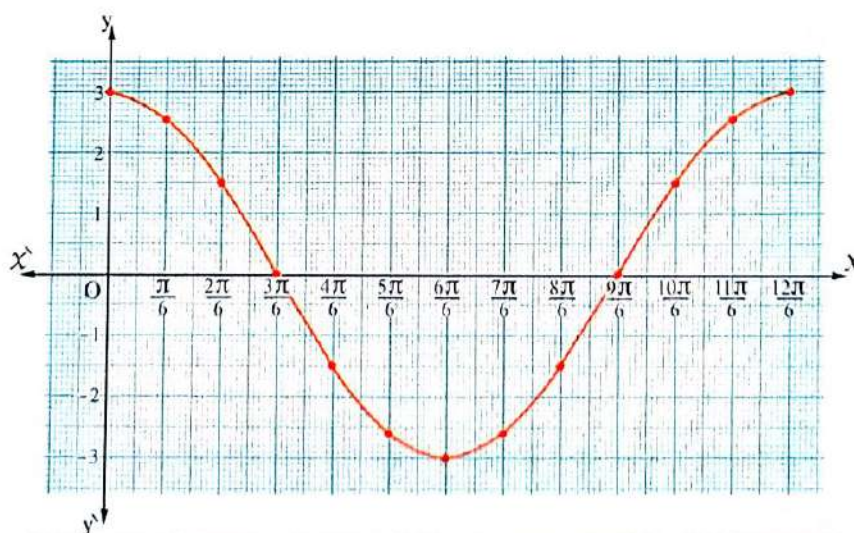
- 1 The domain of the cosine function is $[-\infty, \infty]$
- 2 • The maximum value of the function equals 1 and it happens when $\theta = \pm 2n\pi$, where $n \in \mathbb{Z}$
 • The minimum value of the function equals -1 and it happens when $\theta = \pi \pm 2n\pi$, where $n \in \mathbb{Z}$
- 3 The range of the function = $[-1, 1]$
- 4 The function is periodic and its period is 2π (360°)

Example 2

Graph the function where $y = 3 \cos \theta$, where $\theta \in [0, 2\pi]$, and from the graph find the maximum and minimum values of the function, its range and its period.

Solution

θ	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
y	3	2.6	1.5	0	-1.5	-2.6	-3	-2.6	-1.5	0	1.5	2.6	3



- The maximum value of the function = 3 , the minimum value of the function = - 3
- The range of the function = $[-3, 3]$
- The period of the function = 2π (360°)

TRY TO SOLVE

Represent graphically the function $f : f(\theta) = 2 \cos \theta$, where $\theta \in [0, 2\pi]$, then from the graph find :

- 1 The maximum and minimum values of the function.
- 2 The range of the function.
- 3 The period of the function.

Note


Each of the two functions : $y = a \sin b \theta$, $y = a \cos b \theta$ is periodic , its period is $\frac{2\pi}{|b|}$ and its range is $[-a, a]$ where a is positive.

Example 3

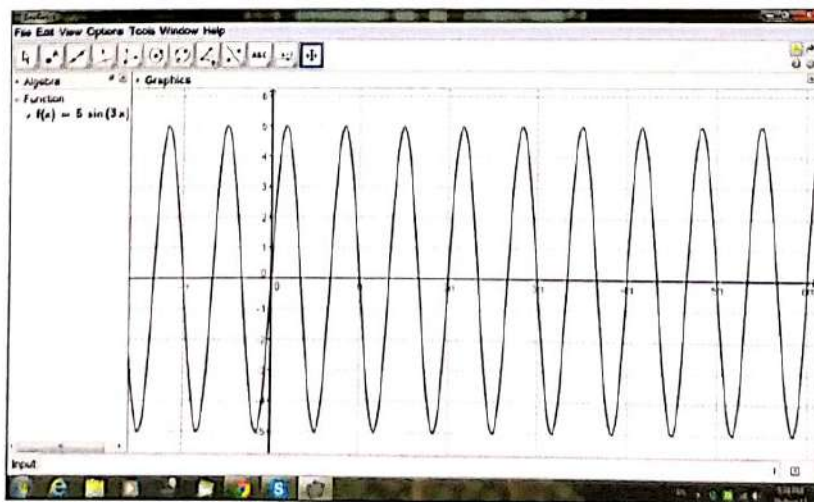
Use a graph program on your computer to graph the function where $y = 5 \sin 3 \theta$, and from the graph , find :

- The range of the function.
- The maximum and minimum values of the function.
- The period of the function.

Solution

We will use Ge  Gebra Program that we can download for free from the website "www.geogebra.org"

- 1 Write in the "input" bar the form of the function " $y = 5 \sin (3 x)$ "
- 2 Press "enter" and the graph will appear as follows :



- The range of the function = $[-5, 5]$
- The maximum value = 5 , the minimum value = -5
- The period of the function = $\frac{2\pi}{|b|} = \frac{2\pi}{3}$ i.e. 120°

Note It is possible to graph the function $y = 5 \sin 3\theta$ (in the previous example) where :
 $0^\circ \leq \theta \leq 120^\circ$ without using the computer as follows :

$$\therefore 0^\circ \leq \theta \leq 120^\circ$$

$$\therefore 0^\circ \leq 3\theta \leq 360^\circ$$

Substituting in 3θ with some values of special angles :

$$0^\circ, 30^\circ, 60^\circ, 90^\circ, 120^\circ, \dots, 360^\circ$$

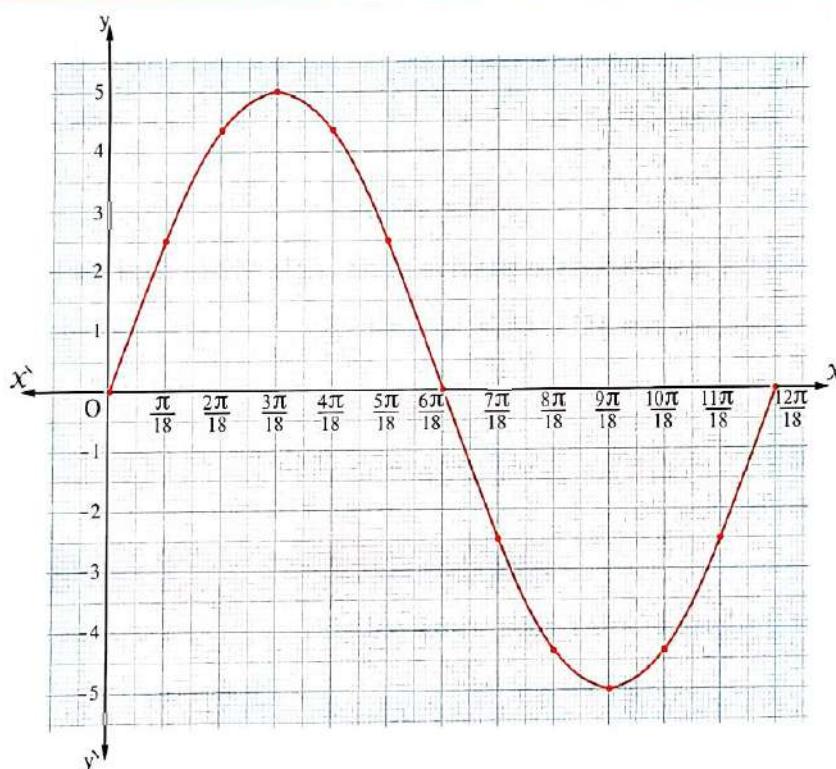
We get the values of θ by dividing by 3 , which are :

$$0^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ, \dots, 120^\circ$$

which is equivalent to : $0, \frac{\pi}{18}, \frac{2\pi}{18}, \frac{3\pi}{18}, \dots, \frac{12\pi}{18}$

Then we form the following table :

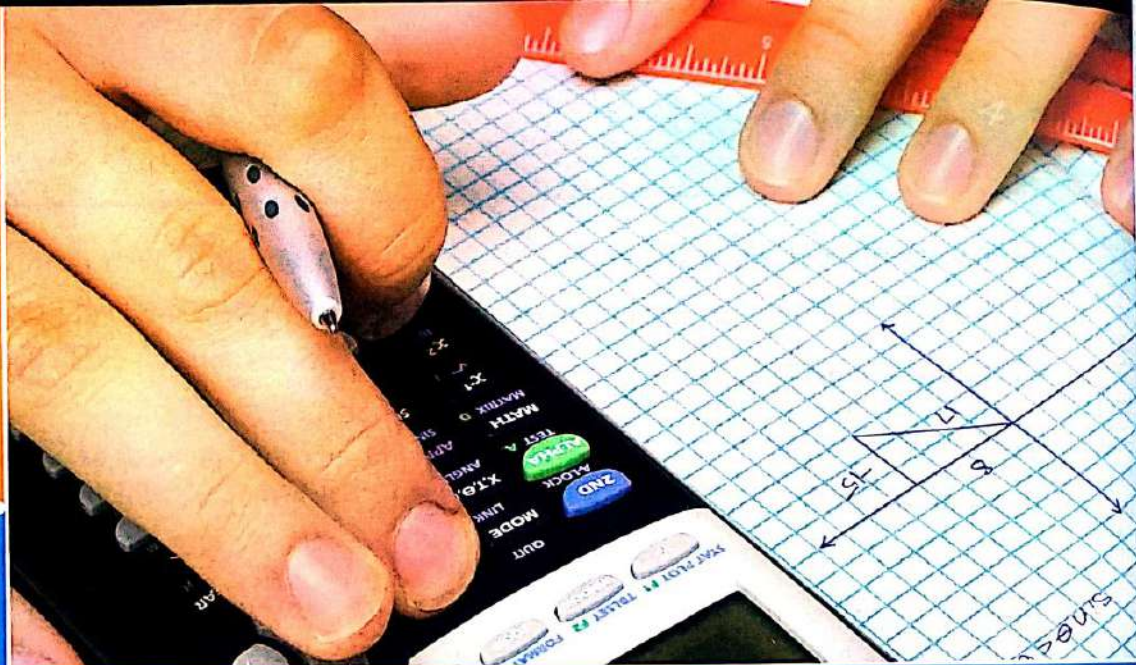
θ	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18}$	$\frac{10\pi}{18}$	$\frac{11\pi}{18}$	$\frac{12\pi}{18}$
$y = 5 \sin 3\theta$	0	2.5	4.3	5	4.3	2.5	0	-2.5	-4.3	-5	-4.3	-2.5	0



The graph represents one period of the function where $y = 5 \sin 3\theta$ which could be repeated to get the graph that appears when we represent it by using computer.

Lesson

6



Finding the measure of an angle given the value of one of its trigonometric ratios

- We know that if $y = \sin \theta$, then we can get the value of y if the value of θ is known.

For example : If $\theta = 30^\circ$, then $y = \sin 30^\circ = \frac{1}{2}$

- There is another form used to find the value of θ if the value of y is known which is $\theta = \sin^{-1} y$ which means that θ equals the value of the measure of the angle whose sine is y

For example : If θ is the measure of a positive acute angle, and :

1 $\theta = \sin^{-1} \frac{1}{2}$

i.e. θ equals the measure of the positive acute angle whose sine = $\frac{1}{2}$

, then $\theta = 30^\circ$, because $\sin 30^\circ = \frac{1}{2}$

2 $\theta = \cos^{-1} \frac{1}{\sqrt{2}}$

i.e. θ equals the measure of the positive acute angle whose cosine = $\frac{1}{\sqrt{2}}$

, then $\theta = 45^\circ$, because $\cos 45^\circ = \frac{1}{\sqrt{2}}$

3 $\theta = \tan^{-1} 1$

i.e. θ equals the measure of the positive acute angle whose tangent = 1

, then $\theta = 45^\circ$, because $\tan 45^\circ = 1$

Example 1

Find the value of θ where $0^\circ < \theta < 360^\circ$ which satisfies :

1 $\theta = \sin^{-1} \frac{\sqrt{3}}{2}$

2 $\theta = \tan^{-1} \sqrt{3}$

3 $\theta = \sec^{-1} \sqrt{2}$

Solution

1 $\therefore \theta = \sin^{-1} \frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2} > 0$ (positive) $\therefore \theta$ lies in the 1st or 2nd quadrant.

\therefore The 1st quadrant : $\theta = 60^\circ$, the second quadrant : $\theta = 180^\circ - 60^\circ = 120^\circ$

2 $\therefore \theta = \tan^{-1} \sqrt{3}$, $\sqrt{3} > 0$ (positive) $\therefore \theta$ lies in the 1st or 3rd quadrant.

\therefore The 1st quadrant : $\theta = 60^\circ$, the 3rd quadrant : $\theta = 180^\circ + 60^\circ = 240^\circ$

3 $\therefore \theta = \sec^{-1} \sqrt{2}$ $\therefore \theta = \cos^{-1} \frac{1}{\sqrt{2}}$

$\therefore \frac{1}{\sqrt{2}} > 0$ (positive)

$\therefore \theta$ lies in the 1st or 4th quadrant.

\therefore The 1st quadrant : $\theta = 45^\circ$

, the 4th quadrant : $\theta = 360^\circ - 45^\circ = 315^\circ$

Notice that

$$\sec^{-1} y = \cos^{-1} \frac{1}{y} \text{ where } y \neq 0$$

TRY TO SOLVE

Find θ where $0^\circ < \theta < 360^\circ$ which satisfies :





1 $\theta = \cos^{-1} \left(\frac{-1}{\sqrt{2}} \right)$

2 $\theta = \csc^{-1} (-2)$

3 $\theta = \cot^{-1} \left(\frac{-1}{\sqrt{3}} \right)$

Note

The functions : $\theta = \sin^{-1} y$, $\theta = \cos^{-1} y$, $\theta = \tan^{-1} y$ are known by the inverse functions of the functions : sine , cosine , tangent , so when using the calculator to evaluate the values of

these functions , it's necessary to use  key before using the three keys  ,  ,  , to get the measure of the angle whose sine or cosine or tangent is given.

Example 2

Find the measure of the positive acute angle θ which satisfies each of the following :

1 $\theta = \sin^{-1} 0.6438$

2 $\theta = \cos^{-1} 0.4517$

Solution

1 Using the keys of the calculator in the following succession from the left :

  0 . 6 4 3 8 = 

, then the number $40^\circ 43' 32.75''$ will appear on the display. $\therefore \theta \approx 40^\circ 43' 33''$

2 Using the keys of the calculator in the following succession from the left :

  0 . 4 5 1 7 = 

, then the number $63^\circ 8' 49.9''$ will appear on the display. $\therefore \theta \approx 63^\circ 8' 50''$

Example 3

If $0^\circ < \theta < 360^\circ$, find θ which satisfies each of the following :

1 $\cos \theta = 0.8177$

2 $\cot \theta = -8.6421$

Solution

1 $\because \cos \theta = 0.8177 > 0$ (positive)

$\therefore \theta$ lies in the 1st or 4th quadrant.

We find firstly $\cos^{-1} 0.8177$ using the keys of the calculator in the following succession from the left :



$\therefore \cos^{-1} 0.8177 \approx 35^\circ 8' 41''$

\therefore The 1st quadrant : $\theta \approx 35^\circ 8' 41''$, the 4th quadrant : $\theta \approx 360^\circ - (35^\circ 8' 41'') = 324^\circ 51' 19''$

2 $\because \cot \theta = -8.6421 < 0$ (negative)

$\therefore \theta$ lies in the 2nd or 4th quadrant.

We find firstly $\cot^{-1} 8.6421$ using the keys of the calculator in the following succession from the left :



$\therefore \cot^{-1} 8.6421 \approx 6^\circ 36' 2''$

\therefore The 2nd quadrant : $\theta \approx 180^\circ - (6^\circ 36' 2'') = 173^\circ 23' 58''$

, the 4th quadrant : $\theta \approx 360^\circ - (6^\circ 36' 2'') = 353^\circ 23' 58''$

TRY TO SOLVE

Find θ where $0^\circ < \theta < 360^\circ$ which satisfies :

1 $\sin \theta = 0.8$

2 $\cot \theta = 0.4695$

3 $\csc \theta = -2.9115$

Example 4

If the terminal side of the positive directed angle of measure θ in its standard position intersects the unit circle at the point $B\left(-\frac{3}{5}, \frac{4}{5}\right)$, find θ where $0^\circ < \theta < 360^\circ$

Solution

\therefore The point $B\left(-\frac{3}{5}, \frac{4}{5}\right)$ lies in the 2nd quadrant.

\therefore The directed angle of measure θ lies in the 2nd quadrant.

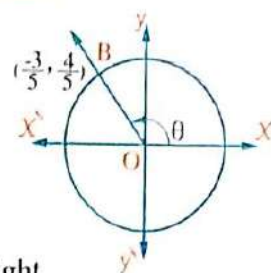
$$\therefore \sin \theta = y = \frac{4}{5} \qquad \therefore \theta = \sin^{-1} \frac{4}{5}$$

and use the keys of the calculator in the following succession from left to right

to find $\sin^{-1} \frac{4}{5}$: 

$$\therefore \sin^{-1} \frac{4}{5} \approx 53^\circ 7' 48''$$

$$\therefore \theta = 180^\circ - (53^\circ 7' 48'') = 126^\circ 52' 12''$$

**Example 5**

A ladder of length 8 m. rests on a vertical wall and a horizontal ground. If the height of the ladder on the ground surface equals 6 m. , find in radian the measure of the angle of inclination of the ladder on the ground.

Solution

The ladder makes with the vertical wall and the horizontal ground a right-angled triangle , let $\triangle ABC$ be right at $\angle C$, $m(\angle CBA) = \theta$

$$\therefore \sin \theta = \frac{AC}{AB} = \frac{6}{8} = \frac{3}{4} \quad , \text{ where } 0^\circ < \theta < 90^\circ$$

$$\therefore \theta = \sin^{-1} \frac{3}{4}$$

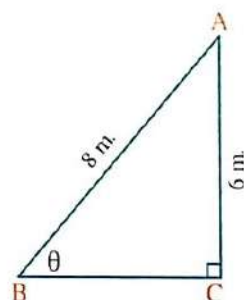
and use the keys of the calculator in the following succession from left to

right to find $\sin^{-1} \frac{3}{4}$: 

$$\therefore \theta \approx 48^\circ 35' 25''$$

$$\therefore \theta^{\text{rad}} = 48^\circ 35' 25'' \times \frac{\pi}{180^\circ} \approx 0.848^{\text{rad}}$$

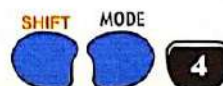
\therefore The measure of the inclination angle of the ladder on the ground $\approx 0.848^{\text{rad}}$

**Note**

In the previous example :

$\theta = \sin^{-1} \frac{3}{4}$, we can get θ in radian directly using the calculator as follows :

- 1 Press , in succession , from left to right to convert the calculator from degree (Deg) system into radian (Rad) system.



2 Find θ in radian directly by pressing in succession from left to right

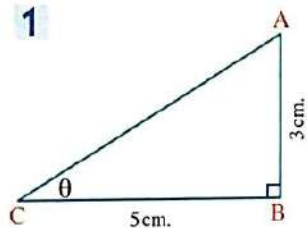
SHIFT \sin^{-1} \sin $\frac{3}{4}$ = 0.00

$\therefore \theta^{\text{rad}} \approx 0.848$

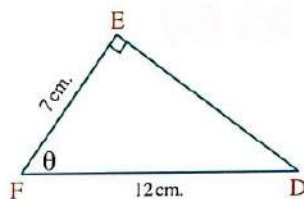
TRY TO SOLVE

Find θ in radian in each of the following right-angled triangles :

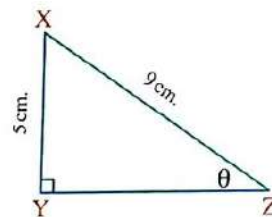
1



2



3



Example 6

If $\sin \theta = \frac{8}{17}$ where $90^\circ < \theta < 180^\circ$, find θ to the nearest second, then find the other trigonometric functions of the angle of measure θ

Solution

$\therefore \sin \theta = \frac{8}{17}$

$\therefore \theta = \sin^{-1} \frac{8}{17} \approx 28^\circ 4' 21''$

$\therefore 90^\circ < \theta < 180^\circ$

$\therefore \theta$ lies in the 2nd quadrant.

$\therefore \theta = 180^\circ - 28^\circ 4' 21'' = 151^\circ 55' 39''$

$\therefore \sin \theta = \frac{8}{17}$

\therefore let $MN = 8$ unit length, $ON = 17$ unit length.

, then (using Pythagoras theorem) $OM = 15$ unit length with a negative sign.

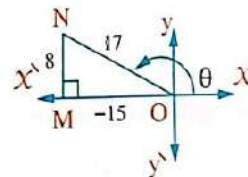
$\therefore \cos \theta = \frac{OM}{ON} = \frac{-15}{17}$

, $\tan \theta = \frac{MN}{OM} = \frac{8}{-15} = -\frac{8}{15}$

, $\csc \theta = \frac{ON}{MN} = \frac{17}{8}$

, $\sec \theta = \frac{ON}{OM} = \frac{17}{-15} = -\frac{17}{15}$

, $\cot \theta = \frac{OM}{MN} = -\frac{15}{8}$



TRY TO SOLVE

If $\sin \theta = \frac{-1}{3}$, $270^\circ < \theta < 360^\circ$

1 Find : θ to the nearest second.

2 Find the value of each of : $\cos \theta$, $\tan \theta$, $\sec \theta$

Example 7

If $\sin \alpha = \frac{3}{5}$ where $90^\circ < \alpha < 180^\circ$, $\tan \beta = \frac{-12}{5}$ where $\beta \in]\frac{3\pi}{2}, 2\pi[$

, $\sin \theta = \sin (180^\circ - \alpha) \cos (\beta - 180^\circ) \cos \alpha$

, find θ to the nearest minute where $0^\circ < \theta < 90^\circ$

Solution

$$\therefore (ON)^2 = (5)^2 - (3)^2 = 16$$

$\therefore ON = 4$ unit length with a negative sign.

$$\therefore (OQ)^2 = (12)^2 + (5)^2 = 169 \quad \therefore OQ = 13 \text{ unit length.}$$

$$\therefore \sin \theta = \sin (180^\circ - \alpha) \cos (\beta - 180^\circ) \cos \alpha$$

$$= \sin \alpha \cos (180^\circ + \beta) \cos \alpha$$

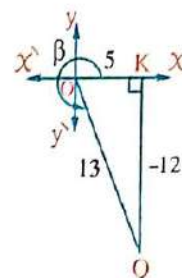
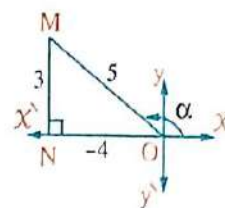
$$= (\sin \alpha) (-\cos \beta) (\cos \alpha)$$

$$= \frac{3}{5} \times \frac{-5}{13} \times \frac{-4}{5} = \frac{12}{65}$$

$$\therefore 0^\circ < \theta < 90^\circ$$

$\therefore \theta$ lies in the 1st quadrant.

Using the calculator, we find that: $\theta \approx 10^\circ 38'$



Example 8

If $5 \sin (180^\circ - \alpha) = 3$ where $0^\circ < \alpha < 90^\circ$, $5 \cot (90^\circ + \beta) - 12 = 0$ where $90^\circ < \beta < 180^\circ$

Find the value of θ where: $\cos \theta = \cos (90^\circ + \alpha) \tan (270^\circ + \beta) \tan (270^\circ - \alpha)$

, where $\theta \in]0, 2\pi[$

Solution

$$\therefore 5 \sin (180^\circ - \alpha) = 3$$

$$\therefore 5 \sin \alpha = 3$$

$$\therefore \sin \alpha = \frac{3}{5} \text{ where } \alpha \text{ lies in the 1}^{\text{st}} \text{ quadrant}$$

$$\therefore 5 \cot (90^\circ + \beta) = 12$$

$$\therefore 5 (-\tan \beta) = 12$$

$$\therefore \tan \beta = \frac{-12}{5} \text{ where } \beta \text{ lies in the 2}^{\text{nd}} \text{ quadrant.}$$

$$\cos \theta = \cos (90^\circ + \alpha) \tan (270^\circ + \beta) \tan (270^\circ - \alpha)$$

$$= (-\sin \alpha) \times (-\cot \beta) \times \cot \alpha$$

$$= \frac{3}{5} \times -\frac{5}{12} \times \frac{4}{3} = -\frac{1}{3}$$

$$\therefore \cos \theta < 0$$

$$\therefore \theta \in \text{the 2}^{\text{nd}} \text{ quadrant}$$

or

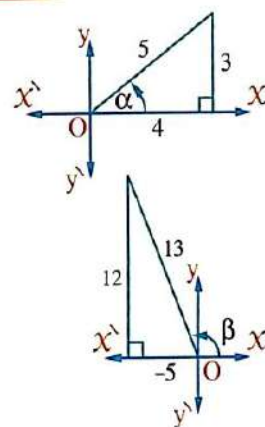
$$\theta \in \text{the 3}^{\text{rd}} \text{ quadrant}$$

$$\therefore \theta = 180^\circ - 70^\circ 32'$$

$$= 109^\circ 28'$$

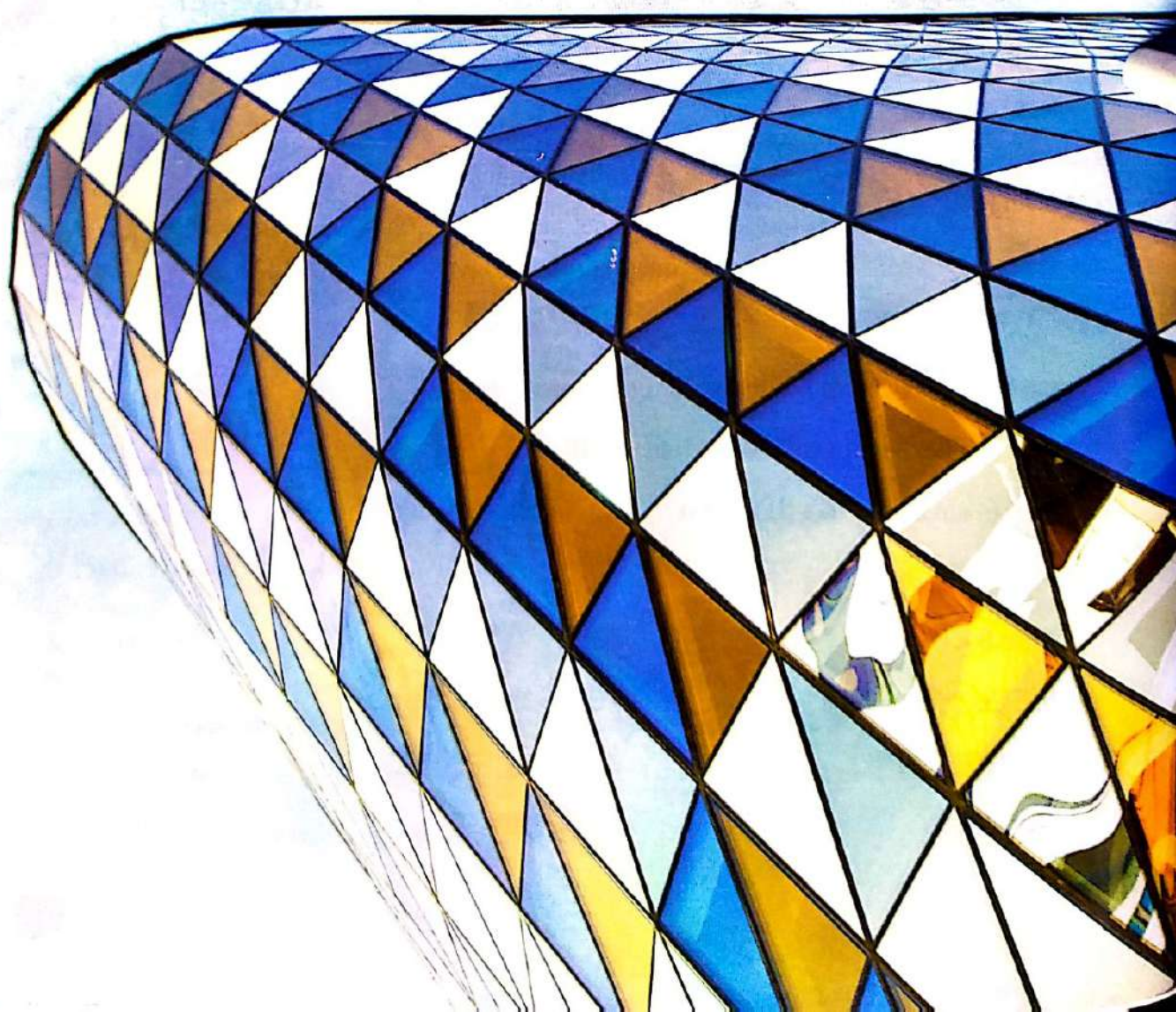
$$\therefore \theta = 180^\circ + 70^\circ 32'$$

$$= 250^\circ 32'$$



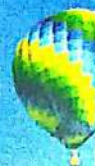
Second

Geometry



UNIT

3



Similarity

Unit Lessons

Lesson 1 : Similarity of polygons.

Lesson 2 : Similarity of triangles.

Lesson 3 : Follow : Similarity of triangles (Theorem (1) - Theorem (2)).

Lesson 4 : The relation between the areas of two similar polygons.

Lesson 5 : Applications of similarity in the circle.

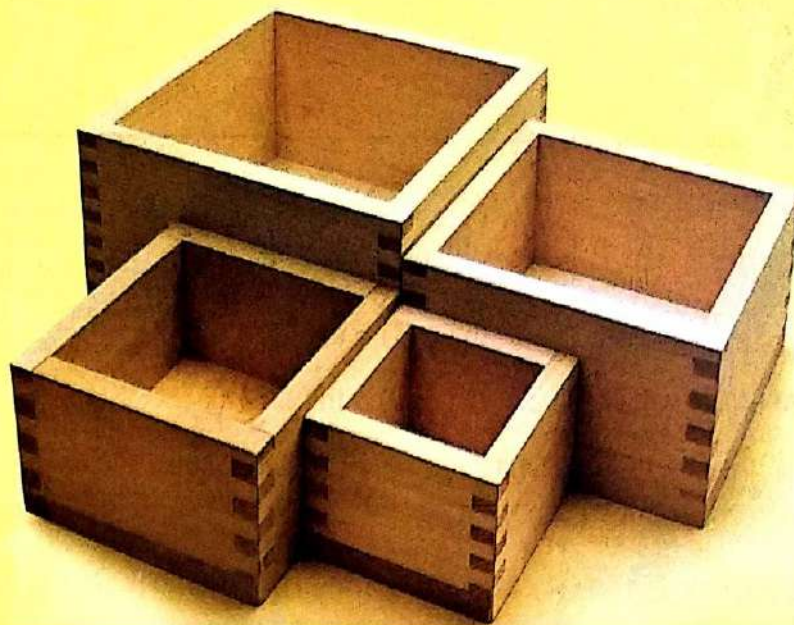
Unit Objectives

By the end of this unit, the student should be able to :

- Revise what he / she has previously studied in the preparatory stage on similarity.
- Use the scale factor of similarity to find lengths of sides of similar polygons.
- Recognize similarity postulate "If two angles of one triangle are congruent to their two corresponding angles of another triangle, then the two triangles are similar".
- Know that : If a line is drawn parallel to one side of a triangle and intersects the other two sides or the lines containing them, then the resulting triangle is similar to the original triangle.
- Know that : In any right-angled triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.
- Solve problems and mathematics applications on cases of similarity of two triangles.
- Recognize and prove the theorem : (If the side lengths of two triangles are in proportion, then the two triangles are similar).
- Recognize and prove the theorem : (If an angle of one triangle is congruent to an angle of another triangle and lengths of the sides including those angles are in proportion, then the triangles are similar).
- Use similarity of triangles in indirect measurements.
- Recognize and prove the theorem : (Ratio of the areas of the surfaces of two similar triangles equals the square of the ratio of the lengths of any two corresponding sides of the two triangles).
- Recognize and prove the theorem : (Ratio of the areas of the surfaces of two similar polygons equals the square of the ratio of the lengths of any two corresponding sides of the two polygons).
- Recognize and deduce the relation between intersecting chords in a circle.
- Recognize and deduce the relation between two secants to a circle from a point outside it.
- Recognize the relation between the length of a tangent to a circle and the two parts of a secant where the tangent and the secant are drawn from the same point outside the circle.
- Model and solve life applications problems by using similarity of polygons in a circle.

Lesson

1



Similarity of polygons

Definition

Two polygons M_1 and M_2 (of same number of sides) are said to be similar if the following two conditions satisfied together :

- 1 Their corresponding angles are congruent.
- 2 The lengths of their corresponding sides are proportional.

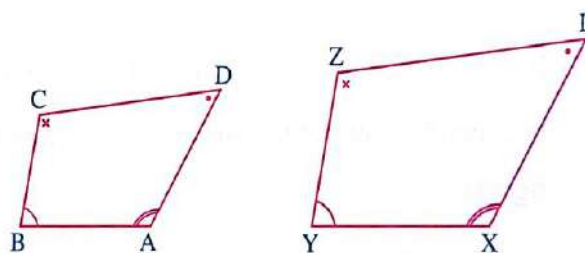
In this case , we shall write :

The polygon $M_1 \sim$ the polygon M_2

That means the polygon M_1 is similar to the polygon M_2

In the opposite figure , if :

- 1 $m(\angle A) = m(\angle X)$
 $, m(\angle B) = m(\angle Y)$
 $, m(\angle C) = m(\angle Z)$
 $, m(\angle D) = m(\angle L)$



- 2 $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX}$

Then the polygon $ABCD \sim$ the polygon $XYZL$

Remark 1

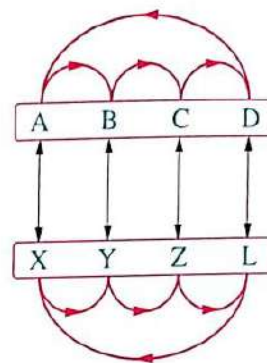
On writing the similar polygons , it is prefer to write them according to the order of their corresponding vertices to make it easy to deduce the equal angles in measure and write the proportion of corresponding side lengths.

For example :

If the polygon $ABCD \sim$ the polygon $XYZL$, then :

$$1 \quad m(\angle A) = m(\angle X) \quad , \quad m(\angle B) = m(\angle Y) \\ , m(\angle C) = m(\angle Z) \quad , \quad m(\angle D) = m(\angle L)$$

$$2 \quad \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX}$$



Remark 2

If the polygon $ABCD \sim$ the polygon $XYZL$, then :

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX} = K \text{ (similarity ratio or scale factor of similarity) , } K > 0$$

If the scale factor of similarity of polygon $ABCD$ to polygon $XYZL = K$

\therefore The scale factor of similarity of polygon $XYZL$ to polygon $ABCD = \frac{1}{K}$

Remark 3

Let K be the similarity ratio of polygon M_1 to polygon M_2 :

- If $K > 1$, then polygon M_1 is an enlargement of polygon M_2 , where K is called the enlargement ratio.
- If $0 < K < 1$, then polygon M_1 is a shrinking to polygon M_2 , where K is called the shrinking ratio.
- If $K = 1$, then polygon M_1 is congruent to polygon M_2

In general , you can use the similarity ratio in calculation of the dimensions of similar figures.

Remark 4

In order that two polygons are similar , the two conditions should be verified together and verifying one of them only is not enough to be similar.

For example :

- All rectangles are not similar because although their corresponding angles are equal in measure (each = 90°) , but the lengths of their corresponding sides may be not proportional.
- Also all rhombuses are not similar because although the lengths of their corresponding sides are proportional , but their corresponding angles may be different in measure.

Remark 5

The congruent polygons are similar but it's not necessary that similar polygons are congruent.

Remark 6

If each of two polygons is similar to a third polygon, then they are similar.

i.e. If polygon $M_1 \sim$ polygon M_3 , polygon $M_2 \sim$ polygon M_3 , then polygon $M_1 \sim$ polygon M_2

Remark 7

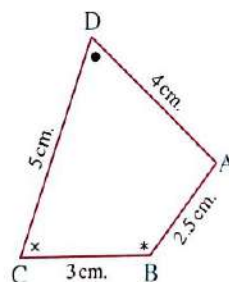
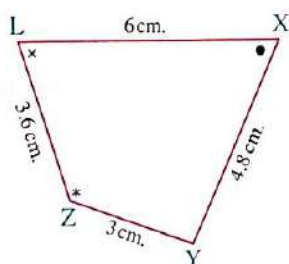
All regular polygons of the same number of sides are similar.

For example : • All equilateral triangles are similar. • All squares are similar.

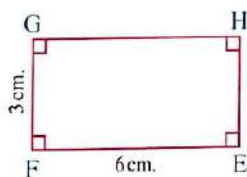
Example 1

Show which of the following pairs of polygons are similar, showing the reason and if they are similar, determine the similarity ratio :

1



2



Solution

1 The two polygons ABCD, YZLX are similar :

Because : $m(\angle B) = m(\angle Z)$, $m(\angle C) = m(\angle L)$, $m(\angle D) = m(\angle X)$

$$\therefore m(\angle A) = m(\angle Y), \frac{AB}{YZ} = \frac{BC}{ZL} = \frac{CD}{LX} = \frac{DA}{XY}, \frac{2.5}{3} = \frac{3}{3.6} = \frac{5}{6} = \frac{4}{4.8}$$

\therefore The similarity ratio = $\frac{5}{6}$

- 2 The two polygons LMNO, EFGH are not similar :

Although : $m(\angle L) = m(\angle E)$, $m(\angle M) = m(\angle F)$, $m(\angle N) = m(\angle G)$
 $m(\angle O) = m(\angle H)$ (Corresponding angles are congruent)

But : $\frac{LM}{EF} \neq \frac{MN}{FG}$

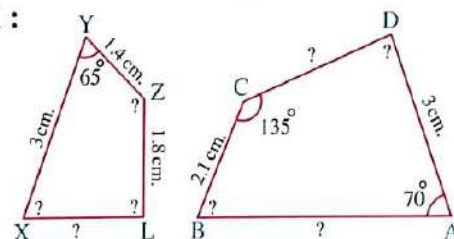
Because : $\frac{8}{6} \neq \frac{5}{3}$

Example 2

In the opposite figure :

If the two polygons ABCD and XYZL are similar, find :

- 1 The scale factor of similarity of polygon ABCD to polygon XYZL
- 2 The lengths of the unknown sides and measures of the unknown angles in each of the two polygons.



Solution

\therefore The polygon ABCD \sim the polygon XYZL

$\therefore \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX}$ = the scale factor of similarity.

$\therefore \frac{AB}{3} = \frac{2.1}{1.4} = \frac{CD}{1.8} = \frac{3}{LX}$ \therefore The scale factor of similarity = $\frac{2.1}{1.4} = \frac{3}{2}$ (First req.)

$\therefore AB = \frac{3 \times 2.1}{1.4} = 4.5 \text{ cm.}$, $CD = \frac{1.8 \times 2.1}{1.4} = 2.7 \text{ cm.}$

, $LX = \frac{1.4 \times 3}{2.1} = 2 \text{ cm.}$

\therefore The polygon ABCD \sim the polygon XYZL

$\therefore m(\angle A) = m(\angle X)$, $m(\angle B) = m(\angle Y)$, $m(\angle C) = m(\angle Z)$

, $m(\angle D) = m(\angle L)$

$\therefore m(\angle X) = 70^\circ$, $m(\angle B) = 65^\circ$, $m(\angle Z) = 135^\circ$

\therefore the sum of measures of the interior angles of a quadrilateral = 360°

$\therefore m(\angle D) = m(\angle L) = 360^\circ - (70^\circ + 65^\circ + 135^\circ) = 90^\circ$ (Second req.)

Remark

In the previous example , we notice that :

\therefore The polygon ABCD \sim the polygon XYZL

$$\therefore \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX} = \text{the scale factor of similarity}$$

$$= \frac{AB + BC + CD + DA}{XY + YZ + ZL + LX} \text{ (from proportion properties)}$$

$$\therefore \frac{\text{Perimeter of the polygon ABCD}}{\text{Perimeter of the polygon XYZL}} = \frac{12.3}{8.2} = \frac{3}{2} = \text{the scale factor of similarity}$$

i.e. The ratio between the perimeters of two similar polygons = the ratio between the lengths of two corresponding sides of them.

Example 3

Two similar polygons , the lengths of sides of one of them are 3 cm. , 5 cm. , 6 cm. , 8 cm. , 10 cm. and the perimeter of the other equals 48 cm. Find the lengths of the sides of the second polygon.

Solution

Let the polygon $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E} \sim$ the polygon ABCDE

$$\therefore \frac{\text{The perimeter of the polygon } \hat{A}\hat{B}\hat{C}\hat{D}\hat{E}}{\text{The perimeter of the polygon ABCDE}} = \frac{\hat{A}\hat{B}}{AB} = \frac{\hat{B}\hat{C}}{BC} = \frac{\hat{C}\hat{D}}{CD} = \frac{\hat{D}\hat{E}}{DE} = \frac{\hat{E}\hat{A}}{EA}$$

$$\therefore \frac{\text{The perimeter of the polygon } \hat{A}\hat{B}\hat{C}\hat{D}\hat{E}}{\text{The perimeter of the polygon ABCDE}} = \frac{48}{3 + 5 + 6 + 8 + 10} = \frac{48}{32} = \frac{3}{2}$$

$$\therefore \frac{\hat{A}\hat{B}}{AB} = \frac{\hat{B}\hat{C}}{BC} = \frac{\hat{C}\hat{D}}{CD} = \frac{\hat{D}\hat{E}}{DE} = \frac{\hat{E}\hat{A}}{EA} = \frac{3}{2}$$

$$\therefore \frac{\hat{A}\hat{B}}{3} = \frac{\hat{B}\hat{C}}{5} = \frac{\hat{C}\hat{D}}{6} = \frac{\hat{D}\hat{E}}{8} = \frac{\hat{E}\hat{A}}{10} = \frac{3}{2}$$

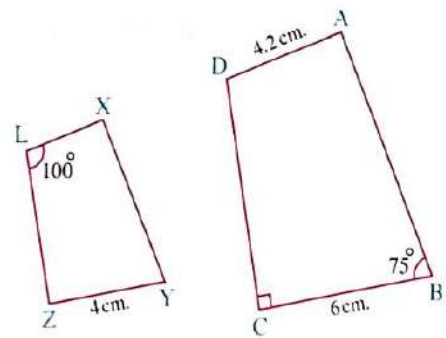
$$\therefore \hat{A}\hat{B} = 4.5 \text{ cm. , } \hat{B}\hat{C} = 7.5 \text{ cm. , } \hat{C}\hat{D} = 9 \text{ cm. , } \hat{D}\hat{E} = 12 \text{ cm. , } \hat{E}\hat{A} = 15 \text{ cm.} \quad (\text{The req.})$$

TRY TO SOLVE

In the opposite figure :

The polygon ABCD ~ the polygon XYZL

- 1 Calculate : $m(\angle X)$, the length of \overline{XL}
- 2 If the perimeter of the polygon ABCD equals 25.8 cm. , calculate the perimeter of the polygon XYZL



Example 4

ABC is a triangle in which : $AB = 4$ cm. , $BC = 5$ cm. , $AC = 8$ cm.

Find the side lengths of another similar triangle if :

- 1 The scale factor of similarity = 2.4
- 2 The scale factor of similarity = 0.7

Solution

- 1 \because The scale factor of similarity = $2.4 > 1$

\therefore The required triangle is an enlargement for $\triangle ABC$

Let $\triangle XYZ \sim \triangle ABC$

$$\therefore \frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = \text{the scale factor of similarity.}$$

$$\therefore \frac{XY}{4} = \frac{YZ}{5} = \frac{ZX}{8} = 2.4$$

$$\therefore XY = 4 \times 2.4 = 9.6 \text{ cm. , } YZ = 5 \times 2.4 = 12 \text{ cm. ,}$$

$$ZX = 8 \times 2.4 = 19.2 \text{ cm.}$$

(The req.)

- 2 \because The scale factor of similarity = $0.7 < 1$

\therefore The required triangle is a shrinking for $\triangle ABC$

Let $\triangle XYZ \sim \triangle ABC$

$$\therefore \frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = \text{the scale factor of similarity.}$$

$$\therefore \frac{XY}{4} = \frac{YZ}{5} = \frac{ZX}{8} = 0.7$$

$$\therefore XY = 4 \times 0.7 = 2.8 \text{ cm. , } YZ = 5 \times 0.7 = 3.5 \text{ cm. , } ZX = 8 \times 0.7 = 5.6 \text{ cm.} \quad (\text{The req.})$$

Lesson

2



Similarity of triangles

Cases of similarity of triangles

First case

Postulate (A. A. similarity postulate)

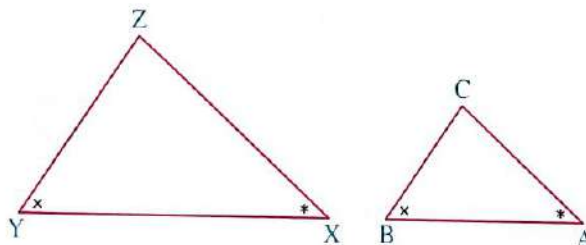
If two angles of one triangle are congruent to their two corresponding angles of another triangle, then the two triangles are similar.

In the opposite figure :

If $\angle A \equiv \angle X$

, $\angle B \equiv \angle Y$

, then $\triangle ABC \sim \triangle XYZ$



Remarks

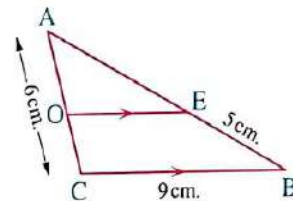
- 1 The two right-angled triangles are similar if the measure of an acute angle in one of them equals the measure of an acute angle in the other.
- 2 The two isosceles triangles are similar if the measure of an angle in one of them equals the measure of the corresponding angle in the other.
- 3 Any two equilateral triangles are similar.

Example 1

In the opposite figure :

ABC is a triangle in which : $AC = \frac{1}{2} AB = 6 \text{ cm.}$

, $BC = 9 \text{ cm.}$, $EB = 5 \text{ cm.}$ and $\overline{EO} \parallel \overline{BC}$



1 Prove that : $\triangle AEO \sim \triangle ABC$

2 Find the length of each of : \overline{EO} and \overline{CO}

Solution

$\therefore \overline{EO} \parallel \overline{BC}$, \overline{AB} is a transversal

, $\therefore \angle A$ is a common angle

$$\therefore \frac{AE}{AB} = \frac{EO}{BC} = \frac{AO}{AC}$$

$$\therefore \frac{1}{2} AB = 6 \text{ cm.}$$

$$\therefore AE = 12 - 5 = 7 \text{ cm.}$$

$$\therefore EO = \frac{9 \times 7}{12} = 5 \frac{1}{4} \text{ cm.}$$

$$\therefore CO = 6 - 3 \frac{1}{2} = 2 \frac{1}{2} \text{ cm.}$$

$$\therefore m(\angle AEO) = m(\angle B) \text{ "corresponding angles"}$$

$$\therefore \triangle AEO \sim \triangle ABC \quad \text{(First req.)}$$

$$\therefore AB = 12 \text{ cm.}$$

$$\therefore \frac{7}{12} = \frac{EO}{9} = \frac{AO}{6}$$

$$\therefore AO = \frac{6 \times 7}{12} = 3 \frac{1}{2} \text{ cm.}$$

(Second req.)

Example 2

\overline{AE} and \overline{BC} are two intersecting chords at D in a circle , where D is the midpoint of \overline{BC}

Prove that : $(BD)^2 = AD \times DE$

Solution

Const. : Draw \overline{AC} and \overline{BE}

Proof : In $\triangle ADC$ and $\triangle BDE$:

$$\therefore m(\angle A) = m(\angle B) \text{ "inscribed angles subtended by } \widehat{CE} \text{ "}$$

$$\therefore m(\angle ADC) = m(\angle BDE) \text{ "V.O.A"}$$

$$\therefore \triangle ADC \sim \triangle BDE$$

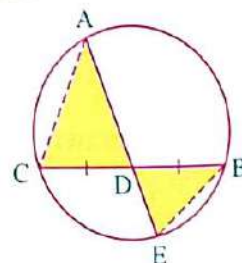
$$\therefore \frac{AD}{BD} = \frac{DC}{DE}$$

$$\therefore BD \times DC = AD \times DE$$

$$\therefore \text{, but } DC = BD \text{ "given"}$$

$$\therefore (BD)^2 = AD \times DE$$

(Q.E.D.)



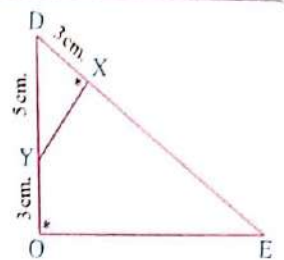
TRY TO SOLVE

In the opposite figure :

DEO is a triangle , $m(\angle O) = m(\angle DXY)$

, $DX = YO = 3$ cm. and $DY = 5$ cm.

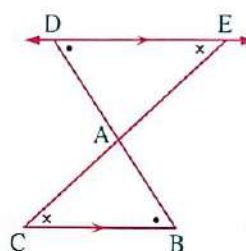
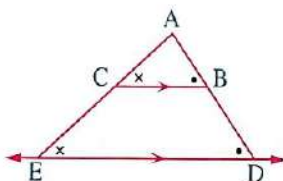
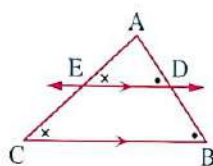
Find the length of : \overline{XE}



Corollary 1

If a line is drawn parallel to one side of a triangle and intersects the other two sides or the lines containing them , then the resulting triangle is similar to the original triangle.

In each of the following figures :



If $\overline{DE} \parallel \overline{BC}$ and intersects \overline{AB} and \overline{AC} at D and E respectively , then $\triangle ABC \sim \triangle ADE$

Example 3

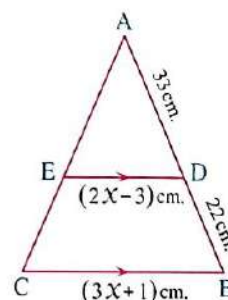
In the opposite figure :

$\overline{DE} \parallel \overline{BC}$, $AD = 33$ cm. , $DB = 22$ cm.

, $DE = (2X - 3)$ cm. and $BC = (3X + 1)$ cm.

1 Prove that : $\triangle ADE \sim \triangle ABC$

2 Find the value of : X



Solution

$$\therefore \overline{DE} \parallel \overline{BC}$$

$$\therefore \triangle ADE \sim \triangle ABC$$

(First req.)

$$\therefore \frac{AD}{AB} = \frac{DE}{BC}$$

$$\therefore \frac{33}{55} = \frac{2X - 3}{3X + 1}$$

$$\therefore \frac{3}{5} = \frac{2X - 3}{3X + 1}$$

$$\therefore 9X + 3 = 10X - 15$$

$$\therefore X = 18$$

(Second req.)

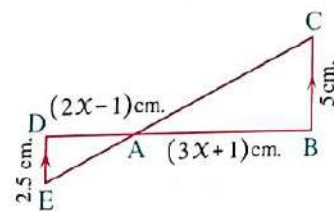
TRY TO SOLVE

In the opposite figure :

$\overline{CE} \cap \overline{BD} = \{A\}$, $\overline{BC} \parallel \overline{DE}$, $BC = 5$ cm. and $DE = 2.5$ cm.

1 Prove that : $\triangle ABC \sim \triangle ADE$

2 Find the value of : x



Corollary 2

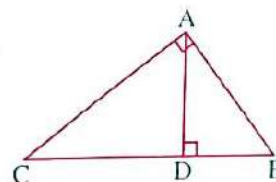
In any right-angled triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.

In the opposite figure :

If $\triangle ABC$ is a right-angled triangle at A and $\overline{AD} \perp \overline{BC}$

, then $\triangle DBA \sim \triangle DAC \sim \triangle ABC$

and it is left to the student to prove this corollary by using the previous postulate and its remarks.



Remarks on the previous figure :

1 From similarity of $\triangle DBA$ and $\triangle ABC$, we get $\frac{DB}{AB} = \frac{BA}{BC}$

$\therefore (AB)^2 = DB \times BC$ i.e. AB is a mean proportional between DB and BC

2 From similarity of $\triangle DAC$ and $\triangle ABC$, we get $\frac{DC}{AC} = \frac{AC}{BC}$

$\therefore (AC)^2 = DC \times BC$ i.e. AC is a mean proportional between DC and BC

3 From similarity of $\triangle DBA$ and $\triangle DAC$, we get $\frac{DA}{DB} = \frac{DB}{DA}$

$\therefore (DA)^2 = DB \times DC$ i.e. DA is a mean proportional between DB and DC

4 From similarity of $\triangle DBA$ and $\triangle ABC$, we get $\frac{AB}{CB} = \frac{AD}{CA}$

$\therefore AD \times CB = AB \times CA$

The previous results are considered as a proof of the Euclidean's theory which we have studied in the preparatory stage.

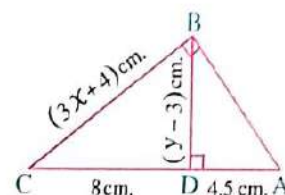
Example 4

In the opposite figure :

ABC is a right-angled triangle at B and $\overline{BD} \perp \overline{AC}$

If AD = 4.5 cm. and DC = 8 cm. ,

find the values of : x and y


Solution

$\therefore \triangle ABC$ is right-angled at B , $\overline{BD} \perp \overline{AC}$

$\therefore \triangle DBC \sim \triangle BAC$

$$\therefore \frac{BC}{AC} = \frac{DC}{BC}$$

$\therefore (BC)^2 = AC \times DC$

$$\therefore (3x + 4)^2 = 12.5 \times 8 = 100$$

$\therefore 3x + 4 = 10$

$$\therefore x = 2$$

$\therefore \triangle ABC$ is right-angled at B , $\overline{BD} \perp \overline{AC}$

$\therefore \triangle ABD \sim \triangle BCD$

$$\therefore \frac{DB}{DC} = \frac{DA}{DB}$$

$\therefore (DB)^2 = DC \times DA$

$$\therefore (y - 3)^2 = 8 \times 4.5 = 36$$

$\therefore y - 3 = 6$

$$\therefore y = 9$$

(The req.)

TRY TO SOLVE

In the opposite figure :

$\triangle ABC$ is right-angled at A , $\overline{AD} \perp \overline{BC}$ Complete :

1 $\frac{BD}{AD} = \frac{AD}{\dots}$

2 $\frac{BD}{AB} = \frac{AD}{\dots}$

3 $\frac{AB}{AC} = \frac{AD}{\dots}$

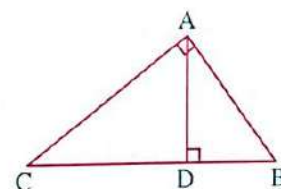
4 $\frac{\dots}{CB} = \frac{AD}{CA}$

5 $\frac{\dots}{AB} = \frac{AB}{\dots}$

6 $(DA)^2 = \dots \times \dots$

7 $(AC)^2 = \dots \times \dots$

8 $AD = \frac{\dots \times CA}{CB}$



Example 5

ABCD is a rectangle, draw $\overline{DF} \perp \overline{AC}$ to cut \overline{AC} in E, \overline{BC} in F

Prove that : The area of the rectangle ABCD = $\sqrt{AE \times AC \times DE \times DF}$

Solution

\therefore ABCD is a rectangle $\therefore m(\angle ADC) = m(\angle BCD) = 90^\circ$

\therefore In $\triangle ADC : m(\angle ADC) = 90^\circ, \overline{DE} \perp \overline{AC}$

$\therefore \triangle ADC \sim \triangle AED$

$$\therefore \frac{AD}{AE} = \frac{AC}{AD} \quad \therefore (AD)^2 = AE \times AC$$

$$\therefore AD = \sqrt{AE \times AC}$$

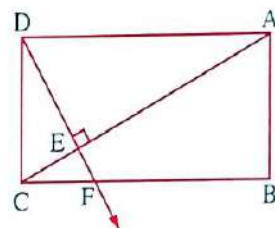
\therefore in $\triangle DCF : m(\angle DCF) = 90^\circ, \overline{CE} \perp \overline{DF}$

$$\therefore \triangle DCF \sim \triangle DEC \quad \therefore \frac{DC}{DE} = \frac{DF}{DC}$$

$$\therefore (DC)^2 = DE \times DF \quad \therefore DC = \sqrt{DE \times DF}$$

$$\therefore \text{The area of the rectangle ABCD} = AD \times DC = \sqrt{AE \times AC} \times \sqrt{DE \times DF}$$

$$= \sqrt{AE \times AC \times DE \times DF} \quad (\text{Q.E.D.})$$



Remember the following relations before solving the exercises :

- The measure of the inscribed angle in a semicircle equals 90°
- In the same circle, the measures of all inscribed angles subtended by the same arc (or subtended by arcs of equal measures) are equal in measures.
- The measure of the angle of tangency is equal to the measure of the inscribed angle subtended by the chord of tangency, in the alternate side.
- In a cyclic quadrilateral :
 - Each two opposite angles are supplementary.
 - The measure of the exterior angle at any vertex is equal to the measure of the interior angle at the opposite vertex.

Lesson

3



Follow : Similarity of triangles (Theorem (1) - Theorem (2))

Second case

Theorem 1 S.S.S. similarity theorem

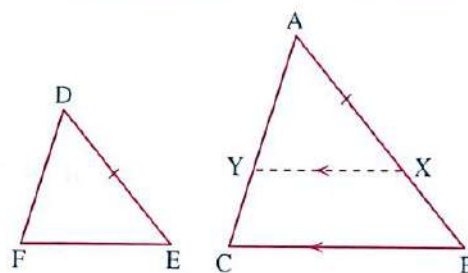
If the side lengths of two triangles are in proportion, then the two triangles are similar.

► **Given** In $\triangle ABC, DEF : \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

► **R.T.P.** $\triangle ABC \sim \triangle DEF$

► **Const.** Take $X \in \overline{AB}$, where $AX = DE$

Draw $\overline{XY} \parallel \overline{BC}$ and intersects \overline{AC} at Y



► **Proof** $\therefore \overline{XY} \parallel \overline{BC} \quad \therefore \triangle ABC \sim \triangle AXY$ "corollary « 1 »"

$\therefore \frac{AB}{AX} = \frac{BC}{XY} = \frac{CA}{YA} \quad , \therefore AX = DE$ "construction"

$$\therefore \frac{AB}{DE} = \frac{BC}{XY} = \frac{CA}{YA} \quad (1)$$

$$, \therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} \quad \text{"given"} \quad (2)$$

From (1), (2) we deduce that : $XY = EF, YA = FD$

and $\triangle AXY \cong \triangle DEF$ "S.S.S. congruency theorem"

$\therefore \triangle DEF \sim \triangle AXY$

$, \therefore \triangle ABC \sim \triangle AXY$ "proved"

$\therefore \triangle ABC \sim \triangle DEF$ (Q.E.D.)

Remark

For writing the two similar triangles in the same order of their corresponding vertices from the proportionality of their side lengths, we follow the following :

Let the vertices of one of the two triangles be A, B and C and the vertices of the other triangle be D, E and F and we have the proportion : $\frac{AC}{DF} = \frac{AB}{EF} = \frac{BC}{DE}$

We search for the vertices of the triangle which are opposite to the sides \overline{AC} , \overline{AB} and \overline{BC} respectively which are B, C and A

and we search for the vertices of the triangle which are opposite to the sides \overline{DF} , \overline{EF} and \overline{DE} respectively which are E, D and F, then :

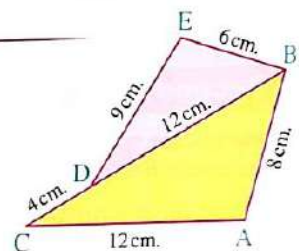
$\Delta BCA \sim \Delta EDF$ or $\Delta ABC \sim \Delta FED$, etc ...

Example 1

In the opposite figure :

Prove that : 1 The two coloured triangles are similar.

2 \overrightarrow{BD} bisects $\angle ABE$



Solution

$$\therefore \frac{AB}{BE} = \frac{8}{9} = \frac{4}{3}, \quad \frac{BC}{BD} = \frac{16}{12} = \frac{4}{3}, \quad \frac{AC}{DE} = \frac{12}{9} = \frac{4}{3}$$

$$\therefore \frac{AB}{BE} = \frac{BC}{BD} = \frac{AC}{DE} \quad \therefore \Delta CAB \sim \Delta DEB$$

(Q.E.D. 1)

From similarity : $m(\angle ABC) = m(\angle EDB)$

$\therefore \overrightarrow{BD}$ bisects $\angle ABE$

(Q.E.D. 2)

Example 2

ABCD is a quadrilateral, $E \in \overline{AC}$, where $\frac{AC}{AD} = \frac{AE}{BE}$ and $\frac{AB}{AE} = \frac{CD}{AC}$

Prove that : 1 $\overline{CD} \parallel \overline{BA}$

2 $\overline{AD} \parallel \overline{BE}$

Solution

$$\therefore \frac{AC}{AD} = \frac{AE}{BE} \quad \therefore \frac{AC}{AE} = \frac{AD}{BE} \quad (1)$$

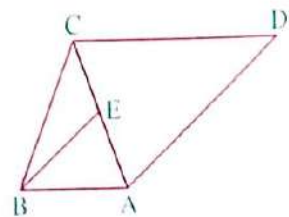
$$\therefore \frac{AB}{AE} = \frac{CD}{AC} \quad \therefore \frac{AC}{AE} = \frac{CD}{AB} \quad (2)$$

From (1), (2) we get : $\frac{AC}{AE} = \frac{AD}{BE} = \frac{CD}{AB}$

$\therefore \Delta DCA \sim \Delta BAE$ we deduce from the similarity that

$m(\angle ACD) = m(\angle EAB)$ and they are alternative angles.

$m(\angle CAD) = m(\angle AEB)$ and they are alternative angles.



$\therefore \overline{CD} \parallel \overline{BA}$ (Q.E.D. 1)

$\therefore \overline{AD} \parallel \overline{BE}$ (Q.E.D. 2)

TRY TO SOLVE

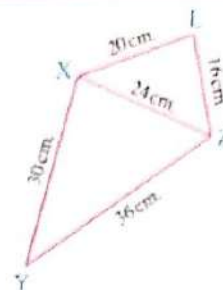
In the opposite figure :

XYZL is a quadrilateral , in which :

$XY = 30$ cm. , $YZ = 36$ cm. , $ZL = 16$ cm.

, $LX = 20$ cm. and $XZ = 24$ cm.

Prove that : $\triangle XYZ \sim \triangle LXZ$



Third case

Theorem 2 S.A.S. similarity theorem

If an angle of one triangle is congruent to an angle of another triangle and lengths of the sides including those angles are in proportion , then the triangles are similar.

► **Given** $\angle A \equiv \angle D$ and $\frac{AB}{DH} = \frac{AC}{DO}$

► **R.T.P.** $\triangle ABC \sim \triangle DHO$

► **Const.** Let $X \in \overline{AB}$ such that $AX = DH$
and draw $\overline{XY} \parallel \overline{BC}$ and intersects \overline{AC} at Y

► **Proof** $\because \overline{XY} \parallel \overline{BC} \therefore \triangle ABC \sim \triangle AXY$ "corollary" (1)

$$\therefore \frac{AB}{AX} = \frac{AC}{AY}$$

$$\because \frac{AB}{DH} = \frac{AC}{DO}$$

$$\therefore AX = DH$$

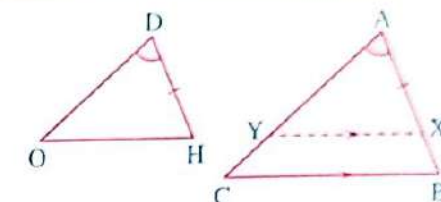
$$\therefore \frac{AB}{AX} = \frac{AC}{DO}$$

$$\therefore AY = DO$$

$$\therefore \triangle AXY \equiv \triangle DHO$$

$$\therefore \triangle AXY \sim \triangle DHO$$

From (1) and (2) we get : $\triangle ABC \sim \triangle DHO$



"given"

"construction"

"S.A.S. congruency theorem"

(2)

(Q.E.D.)

Example 3

ABC is a triangle in which : $AB = 6$ cm. and $BC = 9$ cm. Let D be the midpoint of \overline{AB} and $H \in \overline{BC}$ such that $BH = 2$ cm.

Prove that : 1 $\triangle DBH \sim \triangle CBA$

2 ADHC is a cyclic quadrilateral.

Solution

In $\triangle DBH$ and $\triangle CBA$:

$$\therefore \frac{BH}{BA} = \frac{2}{6} = \frac{1}{3}, \frac{BD}{BC} = \frac{3}{9} = \frac{1}{3}$$

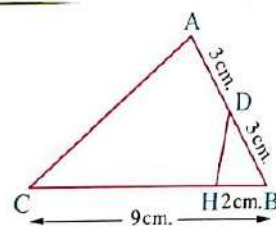
$$\therefore \frac{BH}{BA} = \frac{BD}{BC}, \therefore \angle B \text{ is common.}$$

$$\therefore \triangle DBH \sim \triangle CBA$$

From the similarity of the two triangles, we get that : $m(\angle DHB) = m(\angle A)$

, $\therefore \angle DHB$ is an exterior angle of the quadrilateral ADHC

\therefore The figure ADHC is a cyclic quadrilateral.



(Q.E.D. 1)

(Q.E.D. 2)

Example 4

ABCD is a quadrilateral in which : $m(\angle B) = m(\angle ACD) = 90^\circ$

and $H \in \overline{BC}$ such that : $\frac{CD}{CA} = \frac{BH}{BA}$

Prove that : 1 $\triangle ABH \sim \triangle ACD$

2 $m(\angle AHD) = 90^\circ$

Solution

$$\therefore \frac{CD}{CA} = \frac{BH}{BA}$$

$$\therefore \frac{CD}{BH} = \frac{CA}{BA}$$

$$, \therefore m(\angle B) = m(\angle ACD)$$

$$\therefore \triangle ABH \sim \triangle ACD$$

and hence $m(\angle AHB) = m(\angle ADC)$

, $\therefore \angle AHB$ is an exterior angle of AHCD

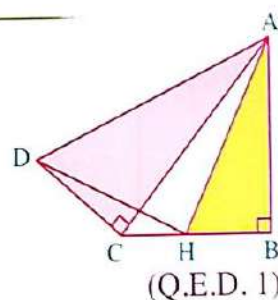
\therefore AHCD is a cyclic quadrilateral.

$$\therefore m(\angle AHD) = m(\angle ACD)$$

$$\therefore m(\angle AHD) = 90^\circ$$

"are drawn on \overline{AD} and on the same side of it"

(Q.E.D. 2)



TRY TO SOLVE

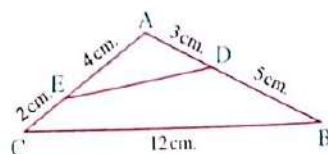
In the opposite figure :

If $AD = 3 \text{ cm.}$, $DB = 5 \text{ cm.}$,

$AE = 4 \text{ cm.}$, $EC = 2 \text{ cm.}$, $BC = 12 \text{ cm.}$

1 Prove that : $\triangle ADE \sim \triangle ACB$

2 Find the length of : \overline{DE}



Lesson

4



The relation between the areas of two similar polygons

- You know that the ratio between the perimeters of two similar polygons equals the ratio between the lengths of any two corresponding sides of them.
- In this lesson you will learn the relation between the areas of two similar polygons.

First The ratio between the areas of the surfaces of two similar triangles

Theorem 3

The ratio between the areas of the surfaces of two similar triangles equals the square of the ratio between the lengths of any two corresponding sides of the two triangles.

► **Given** $\triangle ABC \sim \triangle DHO$

► **R.T.P.** $\frac{\text{The area of } \triangle ABC}{\text{The area of } \triangle DHO} = \left(\frac{AB}{DH}\right)^2 = \left(\frac{BC}{HO}\right)^2 = \left(\frac{AC}{DO}\right)^2$

► **Const.** Draw $\overline{AL} \perp \overline{BC}$ such that :
 $\overline{AL} \cap \overline{BC} = \{L\}$ and $\overline{DM} \perp \overline{HO}$
 such that $\overline{DM} \cap \overline{HO} = \{M\}$

► **Proof** $\therefore \triangle ABC \sim \triangle DHO$

$$\therefore m(\angle B) = m(\angle H) \text{ and } \frac{AB}{DH} = \frac{BC}{HO} = \frac{CA}{OD} \quad (1)$$

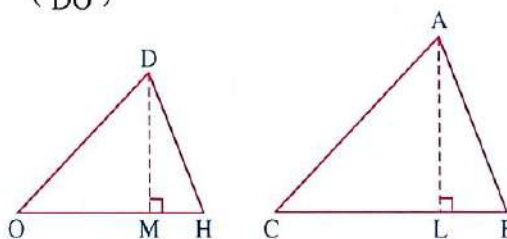
In the two right-angled triangles ABL and DHM : $\therefore m(\angle B) = m(\angle H)$

$$\therefore \triangle ABL \sim \triangle DHM \quad \therefore \frac{AB}{DH} = \frac{AL}{DM} \quad (2)$$

$$\therefore \frac{\text{The area of } \triangle ABC}{\text{The area of } \triangle DHO} = \frac{\frac{1}{2} BC \times AL}{\frac{1}{2} HO \times DM} = \frac{BC}{HO} \times \frac{AL}{DM} \quad (3)$$

From (1), (2) and (3) we get :

$$\frac{\text{The area of } \triangle ABC}{\text{The area of } \triangle DHO} = \frac{BC}{HO} \times \frac{BC}{HO} = \left(\frac{BC}{HO}\right)^2 = \left(\frac{AB}{DH}\right)^2 = \left(\frac{CA}{OD}\right)^2 \quad (\text{Q.E.D.})$$



Example 1

If the ratio between the areas of two similar triangles is $\frac{9}{16}$, the perimeter of the smaller triangle is 60 cm.

Find : The perimeter of the greater triangle.

Solution

Let the two similar triangles be ΔABC , ΔXYZ

$$\therefore \frac{a(\Delta ABC)}{a(\Delta XYZ)} = \left(\frac{AB}{XY}\right)^2 = \frac{9}{16}$$

$$\therefore \frac{AB}{XY} = \frac{3}{4}$$

$$\therefore \frac{\text{The perimeter of } \Delta ABC}{\text{The perimeter of } \Delta XYZ} = \frac{AB}{XY} = \frac{3}{4}$$

$$\therefore \frac{60}{\text{The perimeter of } \Delta XYZ} = \frac{3}{4}$$

$$\therefore \text{The perimeter of } \Delta XYZ = \frac{60 \times 4}{3} = 80 \text{ cm.} \quad (\text{The req.})$$

Example 2

ABC is a triangle of area 62.5 cm^2 . Draw $\overline{XY} \parallel \overline{BC}$ to intersect \overline{AB} at X and \overline{AC} at Y

If $AX : XB = 2 : 3$

Find : The area of the figure XBCY

Solution

In $\Delta ABC : \because \overline{XY} \parallel \overline{BC}$

$\therefore \Delta AXY \sim \Delta ABC$

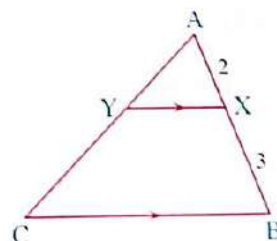
$$\therefore \frac{a(\Delta AXY)}{a(\Delta ABC)} = \left(\frac{AX}{AB}\right)^2$$

$$\therefore \frac{a(\Delta AXY)}{62.5} = \left(\frac{2}{5}\right)^2$$

$$\therefore a(\Delta AXY) = \frac{4}{25} \times 62.5 = 10 \text{ cm}^2$$

$$\therefore \text{The area of the figure XBCY} = a(\Delta ABC) - a(\Delta AXY)$$

$$= 62.5 - 10 = 52.5 \text{ cm}^2 \quad (\text{The req.})$$



Example 3

ABC is a triangle in which : $AB = AC$, $D \in \overline{BC}$, $D \notin \overline{BC}$ and $H \in \overline{CB}$, $H \notin \overline{CB}$

such that $m(\angle BAH) = m(\angle D)$ If the area of ΔACD equals 4 times the area of ΔABH

, then prove that : $DC = 2 AC$

Solution

In $\triangle ABH$ and $\triangle DCA$:

$$\therefore m(\angle BAH) = m(\angle D)$$

$$\text{and } m(\angle ABH) = m(\angle DCA)$$

"Supplementaries of two equal angles in measure"

$$\therefore \triangle ABH \sim \triangle DCA$$

$$\therefore \frac{a(\triangle ABH)}{a(\triangle DCA)} = \left(\frac{AB}{DC}\right)^2$$

$$\therefore \frac{1}{4} = \left(\frac{AB}{DC}\right)^2$$

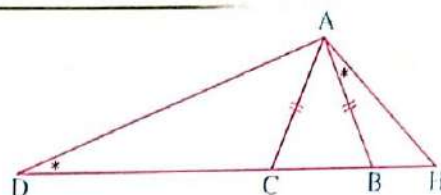
$$\therefore \frac{1}{2} = \frac{AB}{DC}$$

$$\therefore DC = 2AB$$

$$\therefore AB = AC$$

$$\therefore DC = 2AC$$

(Q.E.D.)



Example 4

ABC is a triangle inscribed in a circle such that $\frac{AB}{AC} = \frac{5}{3}$

Draw \overrightarrow{AD} to be a tangent to the circle at A, to intersect \overrightarrow{BC} at D

Find : the area of $\triangle ACD$: the area of $\triangle ABC$

Solution

In $\triangle ADC$ and $\triangle BDA$: $\because \angle D$ is common , $m(\angle CAD) = m(\angle B)$

$$\therefore \triangle ADC \sim \triangle BDA$$

$$\therefore \frac{\text{The area of } \triangle ADC}{\text{The area of } \triangle BDA} = \left(\frac{AC}{BA}\right)^2 = \frac{9}{25}$$

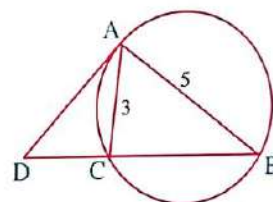
$$\therefore \frac{\text{The area of } \triangle ADC}{\text{The area of } \triangle ABC + \text{The area of } \triangle ADC} = \frac{9}{25}$$

$$\therefore 25 (\text{The area of } \triangle ADC) = 9 (\text{The area of } \triangle ABC) + 9 (\text{The area of } \triangle ADC)$$

$$\therefore 16 (\text{The area of } \triangle ADC) = 9 (\text{The area of } \triangle ABC)$$

$$\therefore \frac{\text{The area of } \triangle ADC}{\text{The area of } \triangle ABC} = \frac{9}{16}$$

(The req.)



TRY TO SOLVE

The ratio between the perimeters of two similar triangles is 4 : 5 If the area of the greater one is 150 cm^2 , find the area of the smaller triangle.

Remark 1

The ratio of the areas of the surfaces of two similar triangles equals the square of the ratio of the lengths of any two corresponding medians of the two triangles.

In the opposite figure :

If $\triangle ABC \sim \triangle DEF$, L is the midpoint of \overline{BC} , M is the midpoint of \overline{EF}

$$\therefore \frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AL}{DM} \right)^2$$

Proof :

$$\because \triangle ABC \sim \triangle DEF$$

$$\therefore BC = 2 BL, EF = 2 EM$$

$$\therefore \frac{AB}{DE} = \frac{BL}{EM}$$

$$\therefore \triangle ABL \sim \triangle DEM$$

$$\therefore \frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AB}{DE} \right)^2$$

$$\text{From (1), (2)} : \therefore \frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AL}{DM} \right)^2$$

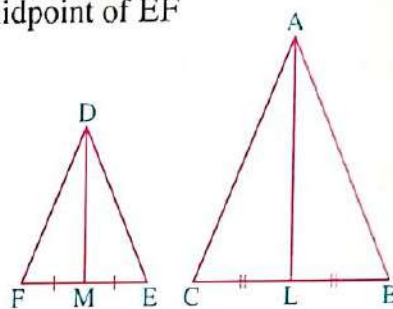
$$\therefore \frac{AB}{DE} = \frac{BC}{EF}$$

$$\therefore \frac{AB}{DE} = \frac{2 BL}{2 EM}$$

$$\therefore \angle B \equiv \angle E \quad (\text{Because } \triangle ABC \sim \triangle DEF)$$

$$\therefore \frac{a(\triangle ABL)}{a(\triangle DEM)} = \left(\frac{AB}{DE} \right)^2 = \left(\frac{AL}{DM} \right)^2 \quad (1)$$

$$(2)$$



Remark 2

In the opposite figure :

If $\triangle ABC \sim \triangle DEF$, \overline{AN} bisects $\angle A$ and intersects \overline{BC} at N

, \overline{DZ} bisects $\angle D$ and intersects \overline{EF} at Z

$$\therefore \frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AN}{DZ} \right)^2$$

Proof :

$$\because \triangle ABC \sim \triangle DEF$$

$$\therefore m(\angle BAC) = m(\angle EDF)$$

$$\therefore \frac{1}{2} m(\angle BAC) = \frac{1}{2} m(\angle EDF)$$

$$\therefore m(\angle BAN) = m(\angle EDZ)$$

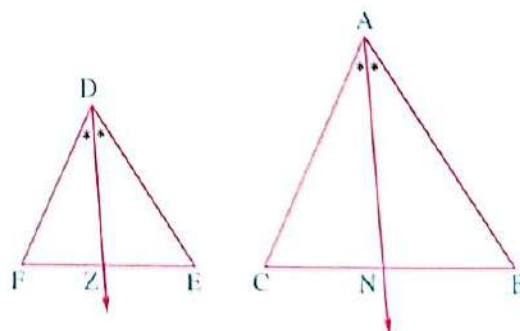
$$\therefore m(\angle B) = m(\angle E)$$

$$\therefore \triangle ABN \sim \triangle DEZ$$

$$\therefore \frac{a(\triangle ABN)}{a(\triangle DEZ)} = \left(\frac{AB}{DE} \right)^2 = \left(\frac{AN}{DZ} \right)^2 \quad (1)$$

$$\therefore \frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AB}{DE} \right)^2 \quad (2)$$

$$\text{From (1), (2)} : \therefore \frac{a(\triangle ABC)}{a(\triangle DEF)} = \left(\frac{AN}{DZ} \right)^2$$



Remark 3

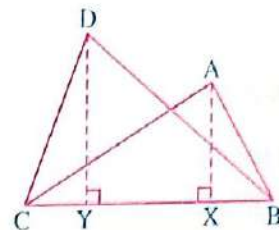
The ratio of the areas of two triangles having a common base equals the ratio of the two heights of the two triangles.

In the opposite figure :

\overline{BC} is a common base of $\triangle ABC$, $\triangle DBC$

$$\therefore \frac{a(\triangle ABC)}{a(\triangle DBC)} = \frac{\frac{1}{2}BC \times AX}{\frac{1}{2}BC \times DY} = \frac{AX}{DY}$$

Notice that : It is not necessary that the two triangles are similar.



Remark 4

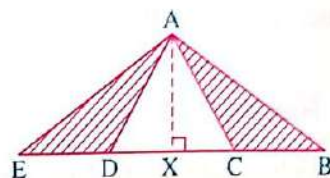
The ratio of the areas of two triangles having a common height equals the ratio of the lengths of two bases of the two triangles.

In the opposite figure :

AX is a common height for $\triangle ABC$, $\triangle ADE$

$$\therefore \frac{a(\triangle ABC)}{a(\triangle ADE)} = \frac{\frac{1}{2}BC \times AX}{\frac{1}{2}DE \times AX} = \frac{BC}{DE}$$

Notice that : It is not necessary that the two triangles are similar.



Example 5

ABC is an inscribed triangle in a circle where $AC > AB$, $D \in \overline{BC}$, where $AD = AB$, draw \overline{AN} a tangent to the circle at A and cuts \overline{CB} at N

Prove that : $BN : DC = (AN)^2 : (CA)^2$

Solution

$$\therefore \frac{a(\triangle ABN)}{a(\triangle CDA)} = \frac{\frac{1}{2}BN \times AX}{\frac{1}{2}DC \times AX} = \frac{BN}{DC} \quad (1)$$

$$\because AB = AD \quad \therefore m(\angle ABD) = m(\angle ADB)$$

$$\therefore m(\angle ABN) = m(\angle ADC)$$

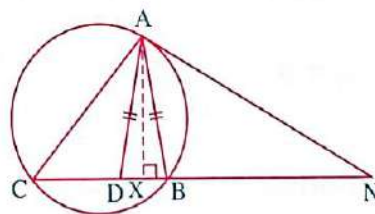
$\because \overline{AN}$ is a tangent.

$$\therefore m(\angle BAN) = m(\angle C) \text{ (drawn on } \widehat{AB} \text{)}$$

$$\therefore \triangle ABN \sim \triangle CDA \quad \therefore \frac{a(\triangle ABN)}{a(\triangle CDA)} = \frac{(AN)^2}{(CA)^2} \quad (2)$$

$$\therefore \text{From (1) and (2)} : \quad \therefore BN : DC = (AN)^2 : (CA)^2$$

(Q.E.D.)



Second The ratio between the areas of the surfaces of two similar polygons

Fact

Any two similar polygons can be divided into the same number of triangles, each is similar to its corresponding one.

In the opposite figure :

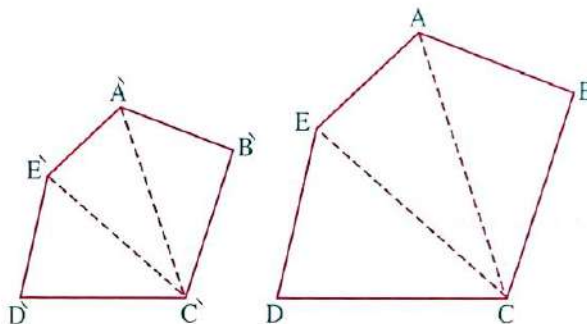
If the two polygons ABCDE and $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}$ are similar and from two corresponding vertices say C

and \hat{C} we draw \overline{CA} , \overline{CE} , \overline{CA}

and $\overline{C\hat{E}}$, then each polygon will

be divided into three triangles

such that : $\triangle ABC \sim \triangle \hat{A}\hat{B}\hat{C}$, $\triangle ACE \sim \triangle \hat{A}\hat{C}\hat{E}$ and $\triangle ECD \sim \triangle \hat{E}\hat{C}\hat{D}$



Remarks

- The previous fact is correct whatever the number of sides of the two similar polygons (having always the same number of sides)
- If the number of sides of a polygon is n sides, then the number of the triangles that the polygon is divided by drawing the diagonals from one of its vertices = $(n - 2)$ triangles

Theorem 4

The ratio between the areas of the surfaces of two similar polygons equals the square of the ratio between the lengths of any two corresponding sides of the polygons.

► **Given**

The polygon ABCDE \sim the polygon $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}$

► **R.T.P.**

$$\frac{a(\text{the polygon ABCDE})}{a(\text{the polygon } \hat{A}\hat{B}\hat{C}\hat{D}\hat{E})} = \left(\frac{AB}{\hat{A}\hat{B}} \right)^2$$

► **Const.**

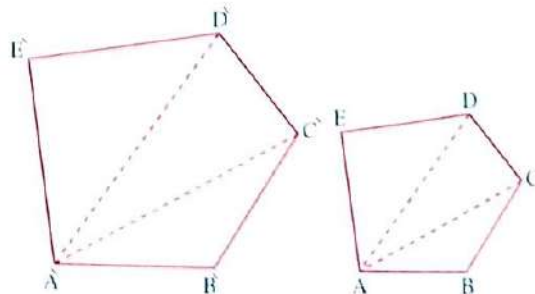
From A, \hat{A} ,

draw \overline{AC} , \overline{AD} , \overline{AC} , \overline{AD}

► **Proof**

\therefore The polygon ABCDE \sim The polygon $\hat{A}\hat{B}\hat{C}\hat{D}\hat{E}$

\therefore They are divided into the same number of triangles each is similar to its corresponding one "fact"



$$\therefore \frac{a(\Delta ABC)}{a(\Delta \tilde{A}\tilde{B}\tilde{C})} = \left(\frac{BC}{\tilde{B}\tilde{C}}\right)^2, \frac{a(\Delta ACD)}{a(\Delta \tilde{A}\tilde{C}\tilde{D})} = \left(\frac{CD}{\tilde{C}\tilde{D}}\right)^2, \frac{a(\Delta ADE)}{a(\Delta \tilde{A}\tilde{D}\tilde{E})} = \left(\frac{DE}{\tilde{D}\tilde{E}}\right)^2$$

$$\therefore \frac{BC}{\tilde{B}\tilde{C}} = \frac{CD}{\tilde{C}\tilde{D}} = \frac{DE}{\tilde{D}\tilde{E}} = \frac{AB}{\tilde{A}\tilde{B}} \text{ "from similar polygons"}$$

$$\therefore \frac{a(\Delta ABC)}{a(\Delta \tilde{A}\tilde{B}\tilde{C})} = \frac{a(\Delta ACD)}{a(\Delta \tilde{A}\tilde{C}\tilde{D})} = \frac{a(\Delta ADE)}{a(\Delta \tilde{A}\tilde{D}\tilde{E})} = \left(\frac{AB}{\tilde{A}\tilde{B}}\right)^2$$

$$\text{From proportion properties : } \frac{a(\Delta ABC) + a(\Delta ACD) + a(\Delta ADE)}{a(\Delta \tilde{A}\tilde{B}\tilde{C}) + a(\Delta \tilde{A}\tilde{C}\tilde{D}) + a(\Delta \tilde{A}\tilde{D}\tilde{E})} = \left(\frac{AB}{\tilde{A}\tilde{B}}\right)^2$$

$$\therefore \frac{a(\text{the polygon } ABCDE)}{a(\text{the polygon } \tilde{A}\tilde{B}\tilde{C}\tilde{D}\tilde{E})} = \left(\frac{AB}{\tilde{A}\tilde{B}}\right)^2 \quad (\text{Q.E.D.})$$

Example 6

The ratio between the perimeters of two similar polygons is 3 : 2

If the sum of their areas is 195 cm^2 , then find the area of each.

Solution

\therefore The ratio between the perimeters is 3 : 2

\therefore The ratio between the lengths of two corresponding sides is 3 : 2

\therefore The ratio between their areas is 9 : 4

Let the area of the first polygon be $9x$ and the area of the second polygon be $4x$

$$\therefore 9x + 4x = 195$$

$$\therefore 13x = 195$$

$$\therefore x = 15$$

$$\therefore \text{The area of the first polygon} = 15 \times 9 = 135 \text{ cm}^2$$

$$\therefore \text{the area of the second polygon} = 15 \times 4 = 60 \text{ cm}^2$$

(The req.)

Example 7

Prove that :

If we construct on the sides of a right-angled triangle, three similar polygons such that the three sides of the triangle correspond to each other, then the area of the polygon constructed on the hypotenuse equals the sum of the areas of the two other polygons.

Solution

\therefore The polygon L \sim the polygon M

$$\therefore \frac{\text{The area of L}}{\text{The area of M}} = \left(\frac{AB}{BC} \right)^2 = \frac{(AB)^2}{(BC)^2} \quad (1)$$

\therefore The polygon N \sim the polygon M

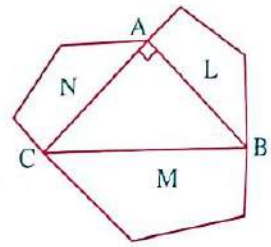
$$\therefore \frac{\text{The area of N}}{\text{The area of M}} = \left(\frac{AC}{BC} \right)^2 = \frac{(AC)^2}{(BC)^2} \quad (2)$$

Adding (1) and (2) : $\therefore \frac{\text{The area of L}}{\text{The area of M}} + \frac{\text{the area of N}}{\text{the area of M}} = \frac{(AB)^2}{(BC)^2} + \frac{(AC)^2}{(BC)^2}$

$$\therefore \frac{\text{The area of L} + \text{the area of N}}{\text{The area of M}} = \frac{(AB)^2 + (AC)^2}{(BC)^2} = \frac{(BC)^2}{(BC)^2} = 1 \text{ "Pythagoras"}$$

\therefore The area of L + the area of N = the area of M

(Q.E.D.)



Example 8

ABCD, $\hat{A}\hat{B}\hat{C}\hat{D}$ are two similar polygons, their diagonals intersect at M, N respectively.

Prove that : $\frac{\text{a (the polygon ABCD)}}{\text{a (the polygon } \hat{A}\hat{B}\hat{C}\hat{D})} = \frac{(BM)^2}{(\hat{B}\hat{N})^2}$

Solution

\therefore The two polygons are similar

$\therefore \triangle ABC \sim \triangle \hat{A}\hat{B}\hat{C}$ and we deduce that : $m(\angle 1) = m(\angle 2)$

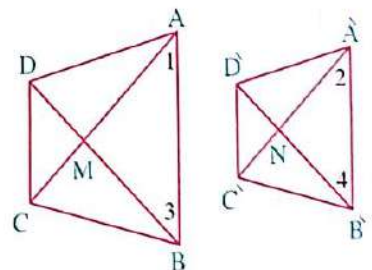
$\therefore \triangle ABD \sim \triangle \hat{A}\hat{B}\hat{D}$ and we deduce that : $m(\angle 3) = m(\angle 4)$

$\therefore \triangle ABM \sim \triangle \hat{A}\hat{B}\hat{N}$

$$\therefore \frac{BM}{\hat{B}\hat{N}} = \frac{AB}{\hat{A}\hat{B}}$$

$$\therefore \frac{\text{a (the polygon ABCD)}}{\text{a (the polygon } \hat{A}\hat{B}\hat{C}\hat{D})} = \frac{(AB)^2}{(\hat{A}\hat{B})^2} = \frac{(BM)^2}{(\hat{B}\hat{N})^2}$$

(Q.E.D.)



TRY TO SOLVE

ABCD, $\hat{A}\hat{B}\hat{C}\hat{D}$ are two similar polygons, X is the midpoint of \overline{BC} , Y is the midpoint of $\overline{\hat{B}\hat{C}}$

Prove that : $\frac{\text{a (the polygon ABCD)}}{\text{a (the polygon } \hat{A}\hat{B}\hat{C}\hat{D})} = \frac{(XD)^2}{(Y\hat{D})^2}$

Lesson

5

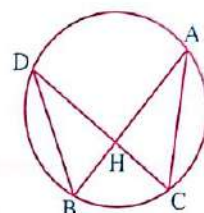
Applications of similarity in the circle

1 In the opposite figure :

\overline{AB} , \overline{CD} are two intersecting chords at H

We notice that : $\triangle AHC \sim \triangle DHB$

because : $m(\angle A) = m(\angle D)$ (two inscribed angles subtended by the same arc \widehat{CB}), $m(\angle AHC) = m(\angle DHB)$ (V.O.A)



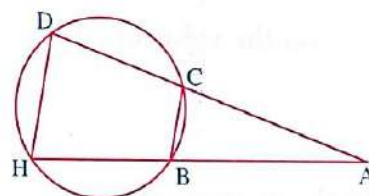
From similarity, we deduce that : $\frac{AH}{DH} = \frac{HC}{HB}$ $\therefore AH \times HB = CH \times HD$

2 In the opposite figure :

BCDH is a quadrilateral, $\overrightarrow{HB} \cap \overrightarrow{DC} = \{A\}$

We notice that : $\triangle ADH \sim \triangle ABC$

because : $m(\angle ABC) = m(\angle D)$ (properties of cyclic quadrilateral), $\angle A$ is a common angle.



From similarity, we deduce that : $\frac{AD}{AB} = \frac{AH}{AC}$ $\therefore AD \times AC = AH \times AB$

Well known problem

If the two lines containing the two chords \overline{AB} , \overline{CD} of a circle intersect at the point E, then $EA \times EB = EC \times ED$

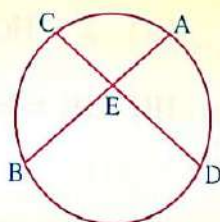


Fig. (1)

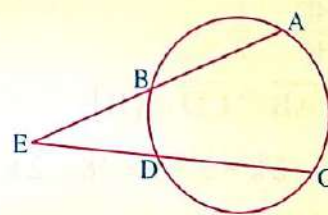


Fig. (2)

Example 1

\overline{AB} and \overline{CD} are two intersecting chords at H in a circle. If $AH = 3$ cm, $HB = 2$ cm, $CD = 5.5$ cm, calculate the length of each of : \overline{CH} , \overline{HD}

Solution

Let $CH = x$ cm.

$$\therefore HD = (5.5 - x) \text{ cm.}$$

$\therefore \overline{AB}$, \overline{CD} are two intersecting chords at H

$$\therefore HA \times HB = HC \times HD$$

$$\therefore 3 \times 2 = x(5.5 - x)$$

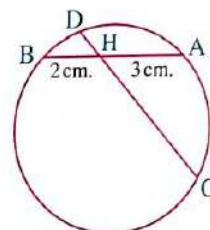
$$\therefore 6 = 5.5x - x^2$$

$$\therefore 2x^2 - 11x + 12 = 0$$

$$\therefore (2x - 3)(x - 4) = 0$$

$$\therefore x = \frac{3}{2} \text{ or } x = 4$$

$$\therefore CH = 4 \text{ cm, } HD = 1.5 \text{ cm.}$$



(The req.)

TRY TO SOLVE

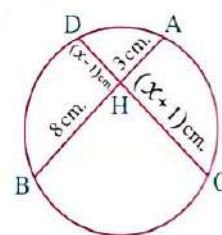
In the opposite figure :

$$\overline{AB} \cap \overline{CD} = \{H\}$$

$$, AH = 3 \text{ cm, } HB = 8 \text{ cm.}$$

$$, CH = (x + 1) \text{ cm, } HD = (x - 1) \text{ cm.}$$

Find the value of : x



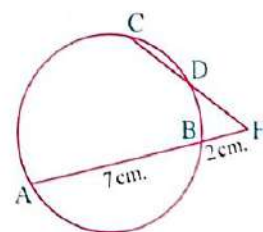
Example 2

In the opposite figure :

$$\overline{AB} \cap \overline{CD} = \{H\}, HB = 2 \text{ cm.}$$

$$, AB = 7 \text{ cm, if } \frac{HD}{HC} = \frac{1}{2}$$

Find the length of : \overline{HC}



Solution

$$\therefore \frac{HD}{HC} = \frac{1}{2}$$

$$\therefore HD = k, HC = 2k \text{ where } k \neq 0$$

$$\therefore \overline{AB} \cap \overline{CD} = \{H\}$$

$$\therefore HD \times HC = HB \times HA$$

$$\therefore k \times 2k = 2 \times 9 = 18 \therefore 2k^2 = 18$$

$$\therefore k^2 = 9$$

$$\therefore k = 3 \text{ or } -3 \text{ (refused)}$$

$$\therefore HC = 2 \times 3 = 6 \text{ cm.}$$

(The req.)

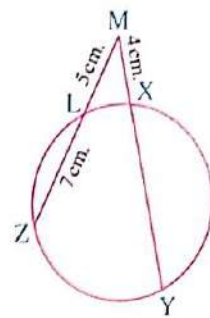
TRY TO SOLVE

In the opposite figure :

$$\overrightarrow{YX} \cap \overrightarrow{ZL} = \{M\}, MX = 4 \text{ cm.}$$

$$, ML = 5 \text{ cm.}, LZ = 7 \text{ cm.}$$

Find the length of : \overline{XY}



In the opposite figure :

\overline{AB} is a tangent to the circle at B

We notice that : $\triangle ABC \sim \triangle ADB$

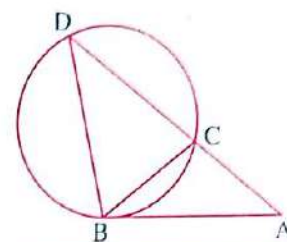
This is because : $m(\angle ABC) = m(\angle D)$

(tangency and inscribed angles subtended by \widehat{BC})

, $\angle A$ is a common angle

From similarity we deduce that :

$$\frac{AB}{AD} = \frac{AC}{AB} \quad \therefore (AB)^2 = AC \times AD$$



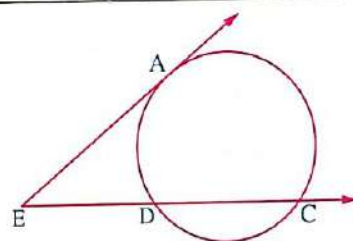
Remember that

AB is a mean proportion of AC, AD

Corollary 1

If E is a point outside the circle, \overrightarrow{EA} is a tangent to the circle at A, \overrightarrow{EC} intersects it at D, C, then

$$(EA)^2 = ED \times EC$$



Example 3

M is a point outside the circle, \overline{MC} is a tangent to the circle at C, \overline{MA} is a secant intersects it at A and B, where $MA > MB$. If $MC = 10 \text{ cm.}$, $AB = 15 \text{ cm.}$

Find the length of : \overline{MB}

Solution

Let $MB = x \text{ cm.}$

$$\therefore MA = (x + 15) \text{ cm.}$$

, $\therefore \overline{MC}$ is a tangent to the circle, \overline{MA} is a secant to it

$$\therefore (MC)^2 = MB \times MA$$

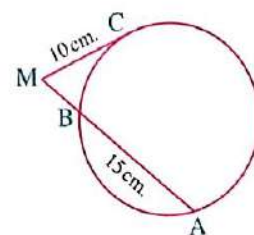
$$\therefore (10)^2 = x(x + 15)$$

$$\therefore x^2 + 15x - 100 = 0$$

$$\therefore (x - 5)(x + 20) = 0$$

$$\therefore x = 5$$

$$\therefore MB = 5 \text{ cm.}$$



(The req.)

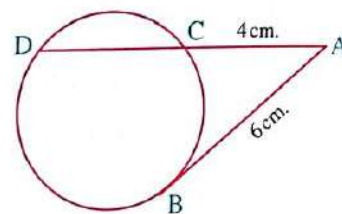
TRY TO SOLVE

In the opposite figure :

\overline{AD} is a secant to the circle at C , D

, \overline{AB} is a tangent to the circle at B

Find the length of : \overline{CD}



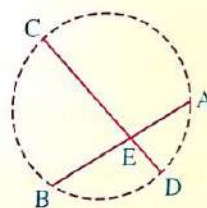
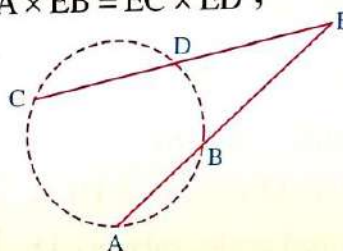
Converse of the well known problem

If the two lines containing the two segments \overline{AB} and \overline{CD} intersect at the point E (A , B , C , D and E are distinct points) and $EA \times EB = EC \times ED$, then the points A , B , C and D lie on a circle.

In the opposite figures :

If $EA \times EB = EC \times ED$

, then the points A , B , C and D lie on the same circle.



Example 4

ABC is a triangle in which : $AC = 9$ cm. , $BC = 12$ cm. Let $D \in \overline{AC}$, where $AD = 5$ cm.

Let $E \in \overline{BC}$, where $\frac{BE}{EC} = 3$

Prove that : The figure ABED is a cyclic quadrilateral.

Solution

$$\therefore CD = AC - AD = 9 - 5 = 4 \text{ cm.}$$

$$\therefore CD \times CA = 4 \times 9 = 36$$

$$\therefore BE = 3 CE$$

$$\therefore BC = 4 CE$$

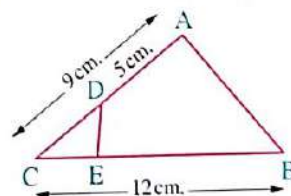
$$\therefore CE = \frac{1}{4} BC = \frac{1}{4} \times 12 = 3 \text{ cm.}$$

$$\therefore CE \times CB = 3 \times 12 = 36$$

$$\therefore CD \times CA = CE \times CB$$

\therefore The figure ABED is a cyclic quadrilateral.

(Q.E.D.)



TRY TO SOLVE

In which of the following figures , do the points A , B , C and D lie on the same circle ?

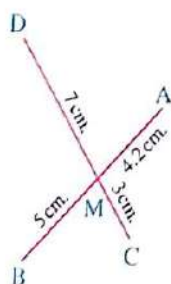


Fig. (1)

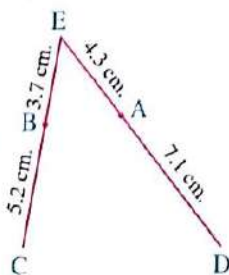


Fig. (2)

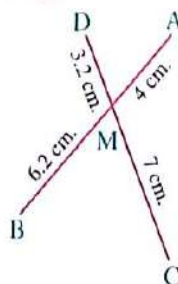


Fig. (3)

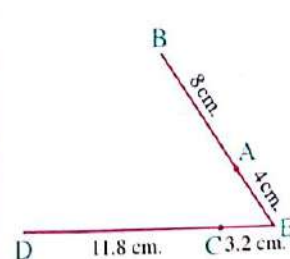
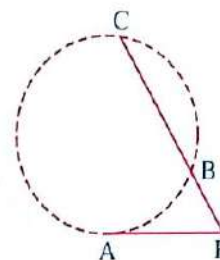


Fig. (4)

Corollary 2

If $(EA)^2 = EB \times EC$
 , then \overline{EA} is a tangent segment to the
 circle which passes through the points
 A , B and C


Example 5

Two intersecting circles at A and B , let $C \in \overrightarrow{BA}$ and $C \notin \overrightarrow{AB}$, let \overline{CD} be a tangent to one of the two circles at D and \overline{CO} intersects the other circle at H and O such that $CO > CH$

Prove that : \overline{CD} is a tangent to the circle passing through D , H and O

Solution

$\therefore \overline{CB}$ and \overline{CO} intersect one of the two circles

$$\therefore CA \times CB = CH \times CO \quad (1)$$

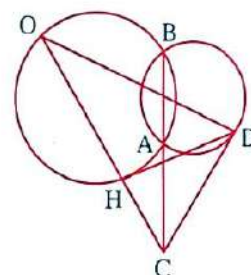
$\therefore \overline{CD}$ is a tangent to the other circle and \overline{CB} intersects it.

$$\therefore (CD)^2 = CA \times CB \quad (2)$$

From (1) and (2) , we get : $(CD)^2 = CH \times CO$

$\therefore \overline{CD}$ is a tangent to the circle passing through D , H and O

(Q.E.D.)


TRY TO SOLVE

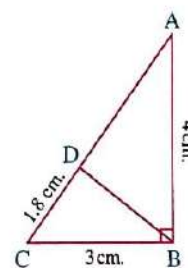
In the opposite figure :

ABC is a right-angled triangle at B

, AB = 4 cm. , BC = 3 cm. , CD = 1.8 cm.

Prove that :

\overline{BC} is a tangent to the circle passing through the points A , B and D



UNIT
4





The Triangle Proportionality Theorems

Unit Lessons

Lesson 1 : Parallel lines and proportional parts.

Lesson 2 : Talis' theorem.

Lesson 3 : Angle bisector and proportional parts.

• Lesson 4 : Follow : Angle bisector and proportional parts (Converse of theorem 3).

Lesson 5 : Applications of proportionality in the circle.

Unit Objectives

By the end of this unit, the student should be able to :

- Recognize and prove the theorem "If a line is drawn parallel to one side of a triangle and intersects the other two sides , then it divides them into segments whose lengths are proportional" and its corollary and its converse.
- Recognize and prove TALIS' general theorem and its special cases.
- Solve problems and mathematical applications on Talis' general theorem and Talis' special theorem.
- Recognize and prove the theorem "The bisector of the interior (or exterior) angle of a triangle at any vertex divides the opposite base ..." and its converse.
- Solve applications about finding the length of each of the interior and the exterior bisectors of an angle of a triangle.
- Recognize the fact "The bisectors of angles of a triangle are concurrent".
- Find the power of a point with respect to a circle.
- Deduce the measures of angles resulting from the intersection of chords and the tangents in a circle.



Preface

Before we study unit 4 (the triangle proportionality theorems)

It is useful and necessary to review the concepts of proportion and some of its properties which will be used in our study in this unit.

- a, b, c, d, e, f, \dots are proportional if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots$

- a, b, c, d, \dots are in continued proportion if $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \dots$

and in this case b is called the middle proportion for a and c , where $b^2 = a \cdot c$

Also, c is called the middle proportion for b and d where $c^2 = b \cdot d$

- If $\frac{a}{b} = \frac{c}{d}$, where a, c are called the antecedents and b, d are called the consequents, then :

- 1 $a \times d = b \times c$

- 2 $\frac{b}{a} = \frac{d}{c}$ (the reciprocal of ratios are equal)

- 3 $\frac{a}{c} = \frac{b}{d}$ $\left(\frac{\text{The antecedent of 1st ratio}}{\text{The antecedent of 2nd ratio}} = \frac{\text{The consequent of 1st ratio}}{\text{The consequent of 2nd ratio}} \right)$

- 4 $\frac{a+b}{b} = \frac{c+d}{d}$ $\left(\frac{\text{antecedent} + \text{consequent}}{\text{consequent}} \text{ of 1st ratio} = \frac{\text{antecedent} + \text{consequent}}{\text{consequent}} \text{ of 2nd ratio} \right)$

- 5 $\frac{a+b}{a} = \frac{c+d}{c}$ $\left(\frac{\text{antecedent} + \text{consequent}}{\text{antecedent}} \text{ of 1st ratio} = \frac{\text{antecedent} + \text{consequent}}{\text{antecedent}} \text{ of 2nd ratio} \right)$

- If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \dots$

, then :

- 1 $\frac{a+c+e+\dots}{b+d+f+\dots} = \text{one of the ratios}$ $\left(\frac{\text{sum of antecedents}}{\text{sum of consequent}} = \text{one of the ratios} \right)$

- 2 $\frac{ka+mc+ne}{kb+md+nf} = \text{one of the ratios}$

, where k, m, n are non zero real numbers

Lesson

1



Parallel lines and proportional parts

Theorem 1

If a line is drawn parallel to one side of a triangle and intersects the other two sides, then it divides them into segments whose lengths are proportional.

► **Given**

ABC is a triangle, $\overline{DE} \parallel \overline{BC}$

► **R.T.P.**

$$\frac{AD}{DB} = \frac{AE}{EC}$$

► **Proof**

$$\therefore \overline{DE} \parallel \overline{BC}$$

$$\therefore \triangle ABC \sim \triangle ADE \text{ "similarity postulate"}$$

$$\therefore \text{then } \frac{AB}{AD} = \frac{AC}{AE} \quad (1)$$

$$\therefore D \in \overline{AB}, E \in \overline{AC}$$

$$\therefore AB = AD + DB, AC = AE + EC \quad (2)$$

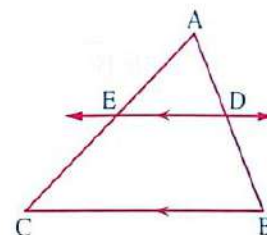
$$\text{From (1), (2) we get: } \frac{AD + DB}{AD} = \frac{AE + EC}{AE}$$

$$\therefore \text{then: } \frac{AD}{AD} + \frac{DB}{AD} = \frac{AE}{AE} + \frac{EC}{AE}$$

$$\therefore 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$$

$$\therefore \frac{DB}{AD} = \frac{EC}{AE}$$

$$\text{From the properties of the proportion, we get: } \frac{AD}{DB} = \frac{AE}{EC}$$



(Q.E.D.)

Remark

From the previous figure :

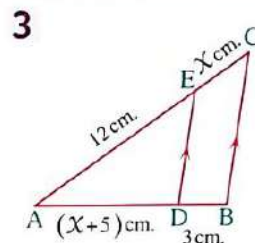
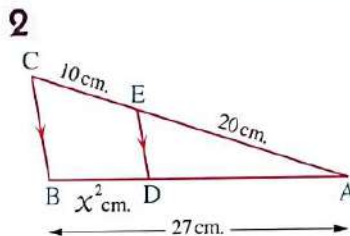
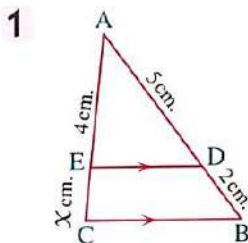
$$\therefore \frac{AD}{DB} = \frac{AE}{EC} \quad \text{"Theorem"}$$

$$\therefore \frac{AD + DB}{DB} = \frac{AE + EC}{EC} \quad (\text{review the proportion properties})$$

$$\therefore \frac{AB}{DB} = \frac{AC}{EC}$$

Example 1

In each of the following figures : $\overline{DE} \parallel \overline{BC}$ Find the value of x

**Solution**

1 $\therefore \overline{DE} \parallel \overline{BC}$

$$\therefore \frac{5}{2} = \frac{4}{x}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

$$\therefore x = 1.6 \text{ cm.}$$

2 $\therefore \overline{DE} \parallel \overline{BC}$

$$\therefore x^2 = 9$$

$$\therefore \frac{AB}{DB} = \frac{AC}{EC}$$

$$\therefore x = 3 \text{ cm.}$$

$$\therefore \frac{27}{x^2} = \frac{30}{10}$$

3 $\therefore \overline{DE} \parallel \overline{BC}$

$$\therefore x^2 + 5x = 36$$

$$\therefore (x+9)(x-4) = 0$$

$$\therefore \frac{AE}{EC} = \frac{AD}{DB}$$

$$\therefore x^2 + 5x - 36 = 0$$

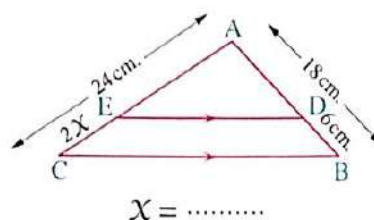
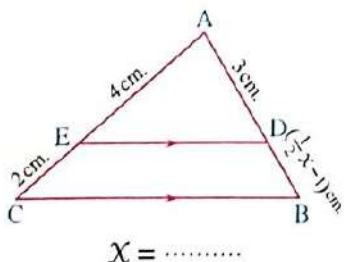
$$\therefore x = -9 \text{ (refused) or } x = 4 \text{ cm.}$$

$$\therefore \frac{12}{x} = \frac{x+5}{3}$$

TRY TO SOLVE

In each of the following figures :

$\overline{DE} \parallel \overline{BC}$, find the numerical value of x



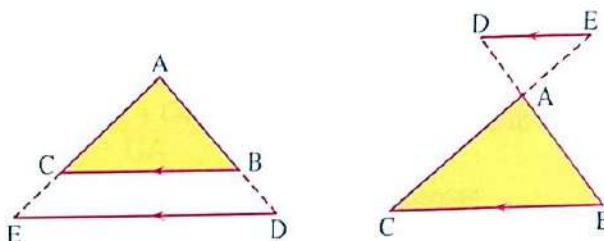
Corollary

If a straight line is drawn outside the triangle ABC parallel to one side of its sides, say \overline{BC} intersecting \overline{AB} and \overline{AC} at D and E respectively, as shown in the figures, then $\frac{AB}{BD} = \frac{AC}{CE}$

From the properties of the proportion

, we can deduce that :

$$\frac{AD}{AB} = \frac{AE}{AC}, \quad \frac{AD}{BD} = \frac{AE}{CE}$$


Example 2

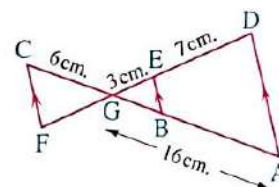
In the opposite figure :

$$\overline{AD} \parallel \overline{EB} \parallel \overline{FC}, \quad \overline{AC} \cap \overline{DF} = \{G\}$$

$$, DE = 7 \text{ cm.}, EG = 3 \text{ cm.}$$

$$, GC = 6 \text{ cm.}, AG = 16 \text{ cm.}$$

Find the length of each of : \overline{GF} and \overline{GB}


Solution

$$\therefore \overline{AD} \parallel \overline{FC}$$

$$\therefore \frac{16}{6} = \frac{10}{GF}$$

$$, \therefore \overline{BE} \parallel \overline{AD}$$

$$\therefore \frac{GB}{16} = \frac{3}{10}$$

$$\therefore \frac{AG}{GC} = \frac{DG}{GF}$$

$$\therefore GF = \frac{6 \times 10}{16} = 3.75 \text{ cm.}$$

$$\therefore \frac{GB}{GA} = \frac{GE}{GD}$$

$$\therefore GB = \frac{3 \times 16}{10} = 4.8 \text{ cm.}$$

(The req.)

TRY TO SOLVE

In the opposite figure :

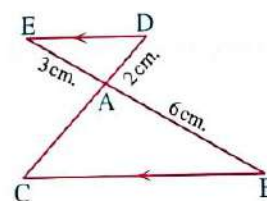
$$\overline{DE} \parallel \overline{BC}, \quad \overline{DC} \cap \overline{BE} = \{A\}$$

$$, AE = 3 \text{ cm.}$$

$$, AB = 6 \text{ cm.}$$

$$\text{and } AD = 2 \text{ cm.}$$

Find the length of \overline{AC}



Converse of theorem 1

If a straight line intersects two sides of a triangle and divides them into segments whose lengths are proportional, then it is parallel to the third side of the triangle.

In the opposite figure :

ABC is a triangle, \overleftrightarrow{DE} intersects \overleftrightarrow{AB} at D

, \overleftrightarrow{AC} at E and $\frac{AD}{DB} = \frac{AE}{EC}$, then $\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$

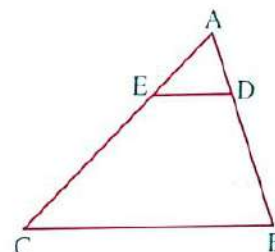
(because $\frac{\text{antecedent} + \text{consequent}}{\text{antecedent}} = \frac{\text{antecedent} + \text{consequent}}{\text{antecedent}}$)

$\therefore \frac{AB}{AD} = \frac{AC}{AE}$, $\because \angle A$ is common.

$\therefore \triangle ABC \sim \triangle ADE$

$\therefore \angle B \equiv \angle ADE$ and they are corresponding angles.

$\therefore \overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$



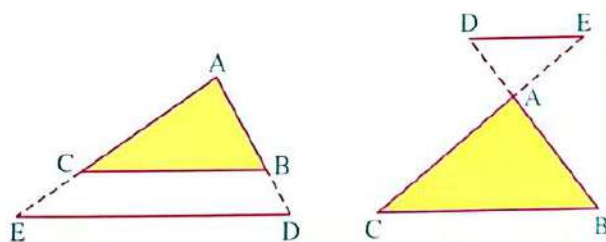
Remark

If a straight line (say \overleftrightarrow{DE}) is drawn outside the triangle ABC, intersecting \overleftrightarrow{AB} and \overleftrightarrow{AC} at D and E respectively

and if $\frac{AD}{DB} = \frac{AE}{EC}$, then $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

In the opposite figures :

If $\frac{AD}{DB} = \frac{AE}{EC}$, then $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

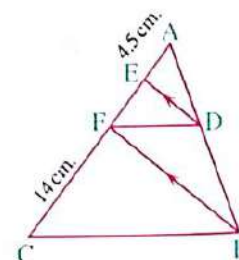


Example 3

In the opposite figure :

If $\overleftrightarrow{DE} \parallel \overleftrightarrow{BF}$, $AD = \frac{3}{4} DB$, $AE = 4.5$ cm., $FC = 14$ cm.

Prove that : $\overleftrightarrow{DF} \parallel \overleftrightarrow{BC}$



Solution

$$\therefore AD = \frac{3}{4} DB$$

$$\therefore \frac{AD}{DB} = \frac{3}{4}$$

$$\therefore \overline{DE} \parallel \overline{BF}$$

$$\therefore EF = \frac{4 \times 4.5}{3} = 6 \text{ cm.}$$

$$\therefore \frac{AF}{FC} = \frac{AD}{DB}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EF}$$

$$\therefore AF = 4.5 + 6 = 10.5 \text{ cm}$$

$$\therefore \overline{DF} \parallel \overline{BC}$$

$$\therefore \frac{3}{4} = \frac{4.5}{EF}$$

$$\therefore \frac{AF}{FC} = \frac{10.5}{14} = \frac{3}{4}$$

(Q.E.D.)

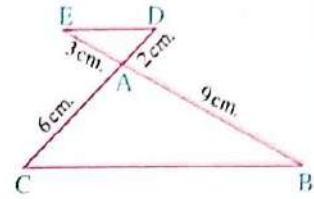
TRY TO SOLVE

In the opposite figure :

$\overline{DC} \cap \overline{BE} = \{A\}$, $AD = 2 \text{ cm.}$, $AE = 3 \text{ cm.}$

, $AB = 9 \text{ cm.}$ and $AC = 6 \text{ cm.}$

Determine whether $\overline{DE} \parallel \overline{BC}$ and why ?



Example 4

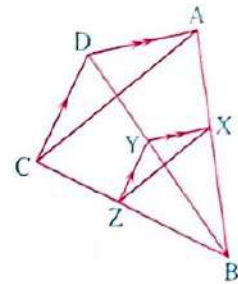
In the opposite figure :

ABCD is a quadrilateral , $Y \in \overline{BD}$, \overline{YX} is drawn

such that $\overline{YX} \parallel \overline{DA}$ intersecting \overline{AB} at X

, \overline{YZ} is drawn such that $\overline{YZ} \parallel \overline{DC}$ intersecting \overline{BC} at Z

Prove that : $\overline{XZ} \parallel \overline{AC}$



Solution

In $\triangle ABD$: $\therefore \overline{XY} \parallel \overline{AD}$

$$\therefore \frac{BX}{BA} = \frac{BY}{BD} \quad (1)$$

In $\triangle BCD$: $\therefore \overline{YZ} \parallel \overline{CD}$

$$\therefore \frac{BZ}{BC} = \frac{BY}{BD} \quad (2)$$

From (1) , (2) : $\therefore \frac{BX}{BA} = \frac{BZ}{BC}$

\therefore In $\triangle ABC$: $\overline{XZ} \parallel \overline{AC}$

(Q.E.D.)

TRY TO SOLVE

In the opposite figure :

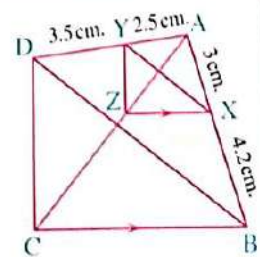
ABCD is a quadrilateral , its diagonals \overline{AC} and \overline{BD} are drawn

, $X \in \overline{AB}$ such that $AX = 3 \text{ cm.}$, $XB = 4.2 \text{ cm.}$, $Y \in \overline{AD}$

such that $AY = 2.5 \text{ cm.}$, $YD = 3.5 \text{ cm.}$

, draw $\overline{XZ} \parallel \overline{BC}$ to intersect \overline{AC} at Z

Prove that : 1 $\overline{XY} \parallel \overline{BD}$ 2 $\overline{YZ} \parallel \overline{CD}$



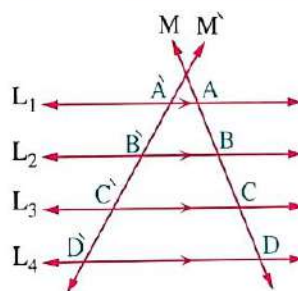
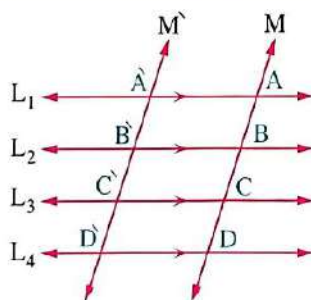
Lesson

2

Talis' theorem

Theorem 2

Given several coplanar parallel lines and two transversals, then the lengths of the corresponding segments on the transversals are proportional.



In the above two figures :

If $L_1 \parallel L_2 \parallel L_3 \parallel L_4$ and M, M' are two transversals, then $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{AC}{A'C'}$

In the following the proof of the theorem

► **Given** $L_1 \parallel L_2 \parallel L_3 \parallel L_4$ and M, M' are two transversals to them

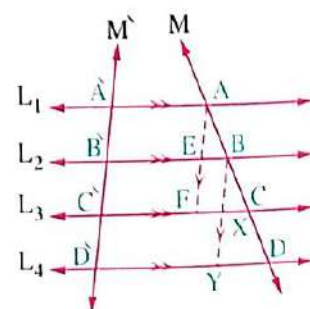
► **R.T.P.** $AB : BC : CD = A'B' : B'C' : C'D'$

► **Const.** Draw $\overrightarrow{AF} \parallel M'$ and intersects L_2 at E ,
 L_3 at F , $\overrightarrow{BY} \parallel M'$ and intersects L_3 at X , L_4 at Y

► **Proof** $\therefore \overrightarrow{AA'} \parallel \overrightarrow{EB'}$, $\overrightarrow{AE} \parallel \overrightarrow{A'B'}$

$\therefore AEB'A'$ is a parallelogram, then $AE = A'B'$

Similarly : $EF = B'C'$, $BX = B'C'$, $XY = C'D'$



In $\triangle ACF$:

$$\therefore \overline{BE} \parallel \overline{CF} \quad \therefore \frac{AB}{BC} = \frac{AE}{EF}$$

$$\text{, then } \frac{AB}{BC} = \frac{\hat{A}\hat{B}}{\hat{B}\hat{C}} \quad , \quad \frac{AB}{\hat{A}\hat{B}} = \frac{BC}{\hat{B}\hat{C}} \quad (\text{exchange the means}) \quad (1)$$

$$\text{Similarly } \triangle BDY : \therefore \frac{BC}{CD} = \frac{\hat{B}\hat{C}}{\hat{C}\hat{D}} \quad , \quad \frac{BC}{\hat{B}\hat{C}} = \frac{CD}{\hat{C}\hat{D}} \quad (\text{exchange the means}) \quad (2)$$

From (1) , (2) we get :

$$\frac{AB}{\hat{A}\hat{B}} = \frac{BC}{\hat{B}\hat{C}} = \frac{CD}{\hat{C}\hat{D}}$$

$$\therefore AB : BC : CD = \hat{A}\hat{B} : \hat{B}\hat{C} : \hat{C}\hat{D} \quad (\text{Q.E.D.})$$

In the previous figure , notice that :

$$\frac{AC}{CD} = \frac{\hat{A}\hat{C}}{\hat{C}\hat{D}} \quad , \quad \frac{AC}{CB} = \frac{\hat{A}\hat{C}}{\hat{C}\hat{B}} \quad , \quad \frac{BD}{DA} = \frac{\hat{B}\hat{D}}{\hat{D}\hat{A}}$$

For example :

In the opposite figure :

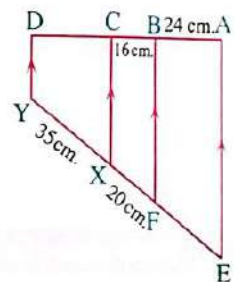
If $\overline{AE} \parallel \overline{BF} \parallel \overline{CX} \parallel \overline{DY}$

such that $AB = 24 \text{ cm.}$, $BC = 16 \text{ cm.}$

, $FX = 20 \text{ cm.}$, $XY = 35 \text{ cm.}$

$$\text{, then } \frac{AB}{EF} = \frac{BC}{FX} = \frac{CD}{XY} \quad \text{i.e.} \quad \frac{24}{EF} = \frac{16}{20} = \frac{CD}{35}$$

$$\text{, then } EF = \frac{20 \times 24}{16} = 30 \text{ cm.} \quad , \quad CD = \frac{16 \times 35}{20} = 28 \text{ cm.}$$



Example 1

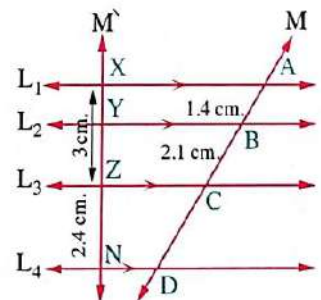
In the opposite figure :

$L_1 \parallel L_2 \parallel L_3 \parallel L_4$ and

M, \hat{M} are two transversals.

Use the lengths shown to

calculate the length of each of \overline{XY} and \overline{CD}



Solution

$\therefore L_1 \parallel L_2 \parallel L_3 \parallel L_4$ and M, \hat{M} are two transversals.

$$\therefore \frac{AB}{XY} = \frac{CD}{ZN} = \frac{AC}{XZ}$$

$$\therefore \frac{1.4}{XY} = \frac{CD}{2.4} = \frac{1.4 + 2.1}{3} = \frac{3.5}{3}$$

$$\therefore XY = \frac{1.4 \times 3}{3.5} = 1.2 \text{ cm. (First req.)}$$

$$\text{, } CD = \frac{2.4 \times 3.5}{3} = 2.8 \text{ cm.}$$

(Second req.)

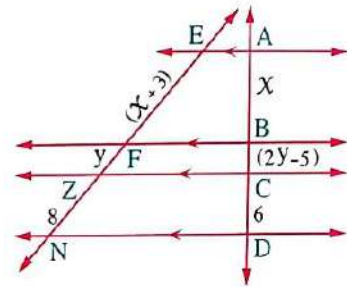
Example 2

In the opposite figure :

If $\overleftrightarrow{AE} \parallel \overleftrightarrow{BF} \parallel \overleftrightarrow{CZ} \parallel \overleftrightarrow{DN}$

Find the numerical value of each of x and y

(lengths are measured in centimetres)



Solution

$\therefore \overleftrightarrow{AE} \parallel \overleftrightarrow{BF} \parallel \overleftrightarrow{CZ} \parallel \overleftrightarrow{DN}$ and \overleftrightarrow{AB} , \overleftrightarrow{EF} are two transversals

$$\therefore \frac{AB}{EF} = \frac{BC}{FZ} = \frac{CD}{ZN}$$

$$\therefore \frac{x}{x+3} = \frac{2y-5}{y} = \frac{6}{8}$$

$$\therefore 8x = 6(x+3)$$

$$\therefore 8x = 6x + 18$$

$$\therefore x = 9$$

$$\therefore 6y = 8(2y-5)$$

$$\therefore 6y = 16y - 40$$

$$\therefore y = 4$$

(The req.)

TRY TO SOLVE

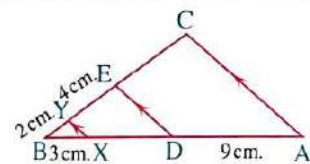
In the opposite figure :

ABC is a triangle ,

$\overleftrightarrow{AC} \parallel \overleftrightarrow{DE} \parallel \overleftrightarrow{XY}$,

AD = 9 cm. , XB = 3 cm. , BY = 2 cm. , EY = 4 cm.

Find : CE and DX



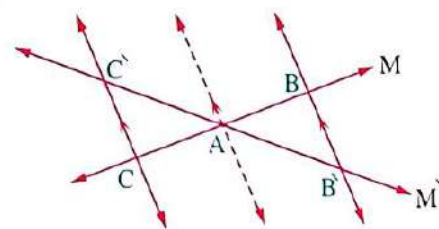
Two special cases

1 If the two lines M and M' intersect at

the point A and $\overleftrightarrow{BB'} \parallel \overleftrightarrow{CC'}$

, then $\frac{AB}{AC} = \frac{AB'}{AC'}$

and conversely if $\frac{AB}{AC} = \frac{AB'}{AC'}$, then $\overleftrightarrow{BB'} \parallel \overleftrightarrow{CC'}$



2 **Talis' special theorem :**

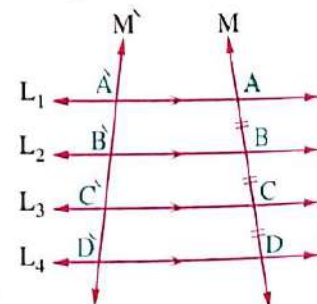
If the lengths of the segments on the transversal are equal , then the lengths of the segments on any other transversal will be also equal.

In the opposite figure :

If $L_1 \parallel L_2 \parallel L_3 \parallel L_4$,

M and M' are two transversals to them

and if $AB = BC = CD$, then $A'B' = B'C' = C'D'$

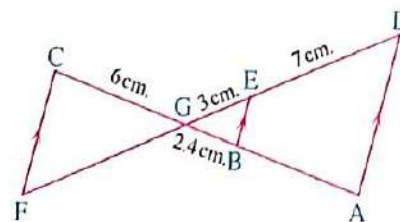


Example 3

In the opposite figure :

$\overline{AD} \parallel \overline{BE} \parallel \overline{FC}$ and \overline{AC} , \overline{DF} are two transversals intersecting at G

Use the shown lengths to calculate the length of each of \overline{GF} , \overline{GA}



Solution

$\therefore \overline{AD} \parallel \overline{BE} \parallel \overline{FC}$ and \overline{AC} , \overline{DF} are two transversals intersecting at G

$$\therefore \frac{GF}{GC} = \frac{GE}{GB} = \frac{GD}{GA}$$

$$\therefore \frac{GF}{6} = \frac{3}{2.4} = \frac{10}{GA}$$

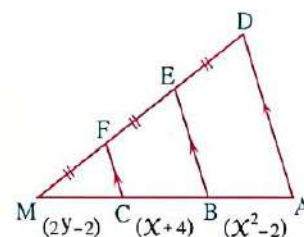
$$\therefore GF = \frac{6 \times 3}{2.4} = 7.5 \text{ cm.} \quad (\text{First req.}) \quad , GA = \frac{2.4 \times 10}{3} = 8 \text{ cm.} \quad (\text{Second req.})$$

Example 4

In the opposite figure :

$\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$, $DE = EF = FM$, find the value of each of x and y

(lengths are measured in centimetres)



Solution

$\therefore \overline{AD} \parallel \overline{BE} \parallel \overline{CF}$, $DE = EF = FM$

$$\therefore AB = BC = CM \quad \therefore x^2 - 2 = x + 4$$

$$\therefore x^2 - x - 6 = 0$$

$$\therefore (x + 2)(x - 3) = 0 \quad \therefore x = -2 \text{ or } x = 3$$

$$\therefore \text{at } x = -2 : \therefore BC = 2 \text{ cm.}$$

$$\therefore \text{at } x = 3 : \therefore BC = 7 \text{ cm.}$$

$$\therefore BC = CM$$

$$\therefore \text{at } BC = 2 \text{ cm.} : \therefore 2y - 2 = 2 \therefore y = 2$$

$$\therefore \text{at } BC = 7 \text{ cm.} : \therefore 2y - 2 = 7$$

$$\therefore y = 4.5$$

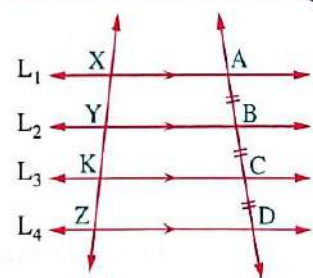
(The req.)

TRY TO SOLVE

In the opposite figure :

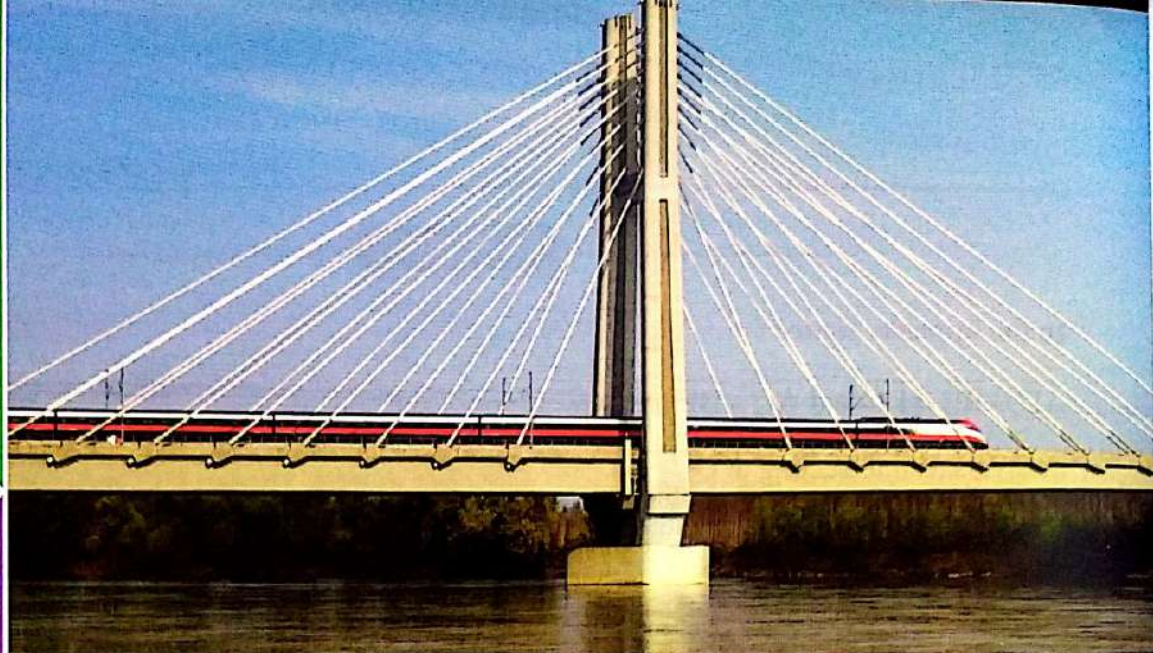
If $XK = 6 \text{ cm.}$

Complete : $YK = \dots\dots\dots \text{ cm.}$



Lesson

3



Angle bisector and proportional parts

Theorem 3

The bisector of the interior or exterior angle of a triangle at any vertex divides the opposite base of the triangle internally or externally into two parts, the ratio of their lengths is equal to the ratio of the lengths of the other two sides of the triangle.

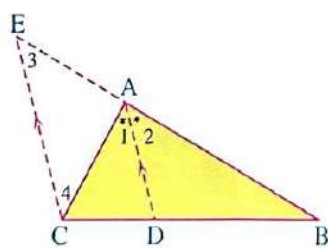


Figure (1)

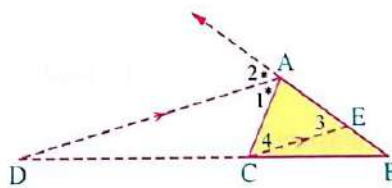


Figure (2)

► **Given** ABC is a triangle, \overrightarrow{AD} bisects $\angle BAC$ internally in figure (1) and externally in figure (2)

► **R.T.P.** $\frac{BD}{DC} = \frac{AB}{AC}$

► **Const.** Draw $\overrightarrow{CE} \parallel \overrightarrow{AD}$ and intersects \overrightarrow{BA} at E

► **Proof** $\because \overrightarrow{AD}$ bisects $\angle BAC$

$$\therefore \angle 1 \equiv \angle 2$$

$$\therefore \overrightarrow{CE} \parallel \overrightarrow{AD}$$

$$\therefore \angle 1 \equiv \angle 4 \text{ (alternate angles)}$$

$$\therefore \angle 3 \equiv \angle 2 \text{ (corresponding angles)}$$

$$\therefore \angle 1 \equiv \angle 2 \quad \therefore \angle 3 \equiv \angle 4$$

$$\therefore \overrightarrow{AE} \equiv \overrightarrow{AC} \quad (1)$$

$$\therefore \overrightarrow{CE} \parallel \overrightarrow{AD}$$

$$\therefore \frac{BD}{DC} = \frac{AB}{AE} \quad (2)$$

$$\text{From (1), (2) : } \therefore \frac{BD}{DC} = \frac{AB}{AC}$$

(Q.E.D.)

Example 1

ABC is a triangle in which $AB = 4 \text{ cm}$, $BC = 5 \text{ cm}$, $CA = 6 \text{ cm}$, draw \overline{AD} to bisect the angle A and intersects \overline{BC} at D

Find the length of each of : \overline{BD} , \overline{DC}

Solution

$\therefore \overline{AD}$ bisects $\angle A$

$$\therefore \frac{BD}{DC} = \frac{BA}{AC}$$

$$\therefore \frac{BD}{DC} = \frac{4}{6} = \frac{2}{3}$$

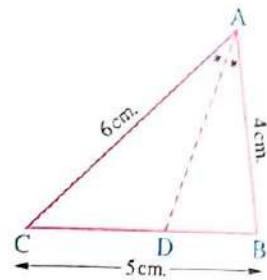
$$\therefore \frac{BD}{5 - BD} = \frac{2}{3}$$

$$\therefore 3 BD = 10 - 2 BD$$

$$\therefore 5 BD = 10$$

$$\therefore BD = 2 \text{ cm}, DC = 5 - 2 = 3 \text{ cm}.$$

(The req.)



Example 2

ABC is a triangle in which $AB = 6 \text{ cm}$, $BC = 5 \text{ cm}$, $CA = 9 \text{ cm}$, draw \overline{AE} to bisect the exterior angle $\angle A$ and intersects \overline{BC} at E

Find the length of each of : \overline{BE} , \overline{EC}

Solution

$\therefore AB < AC$, \overline{AE} bisects the exterior angle at A

$$\therefore E \in \overline{CB}, E \notin \overline{BC}, \frac{BE}{EC} = \frac{BA}{AC}$$

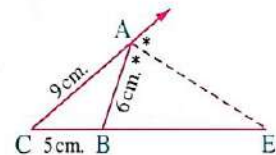
$$\therefore \frac{BE}{EC} = \frac{6}{9} = \frac{2}{3}$$

$$\therefore \frac{BE}{5 + BE} = \frac{2}{3}$$

$$\therefore 3 BE = 10 + 2 BE$$

$$\therefore BE = 10 \text{ cm}, EC = 10 + 5 = 15 \text{ cm}.$$

(The req.)



Example 3

ABC is a triangle, X is the midpoint of \overline{BC} , \overline{XD} bisects $\angle AXB$ and intersects \overline{AB} at D, \overline{XE} bisects $\angle AXC$ and intersects \overline{AC} at E. Prove that : $\overline{DE} \parallel \overline{BC}$

Solution

In $\triangle AXB$: $\therefore \overline{XD}$ bisects $\angle AXB$

$$\therefore \frac{AD}{DB} = \frac{AX}{XB} \quad (1)$$

, in $\triangle AXC$: $\therefore \overline{XE}$ bisects $\angle AXC$

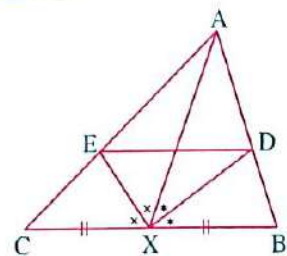
$$\therefore \frac{AE}{EC} = \frac{AX}{XC} \quad (2)$$

From (1), (2) and noticing that : $XB = XC$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

$$\therefore \text{In } \triangle ABC : \overline{DE} \parallel \overline{BC}$$

(Q.E.D.)

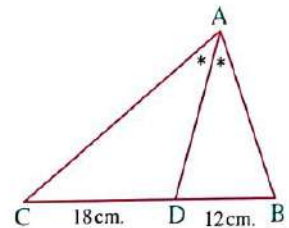


Example 4

In the opposite figure :

ABC is a triangle, \overrightarrow{AD} bisects $\angle A$ and intersects \overline{BC} at D, where
BD = 12 cm., DC = 18 cm., if the perimeter of $\triangle ABC = 80$ cm.

Find the length of each of : \overline{AC} , \overline{AB}



Solution

$$\text{In } \triangle ABC : \because \overrightarrow{AD} \text{ bisects } \angle A \quad \therefore \frac{AB}{AC} = \frac{BD}{DC} = \frac{12}{18} = \frac{2}{3}$$

, \therefore The perimeter of $\triangle ABC = 80$ cm., $BC = 12 + 18 = 30$ cm.

$$\therefore AB + AC = 80 - 30 = 50 \text{ cm.}$$

$$\therefore \frac{AB}{AC} = \frac{2}{3}$$

$$\therefore \frac{AB + AC}{AC} = \frac{2 + 3}{3} \text{ (from the properties of the proportion)}$$

$$\therefore \frac{50}{AC} = \frac{5}{3}$$

$$\therefore AC = \frac{3 \times 50}{5} = 30 \text{ cm.}$$

$$\therefore AB = 50 - 30 = 20 \text{ cm.}$$

(The req.)

TRY TO SOLVE

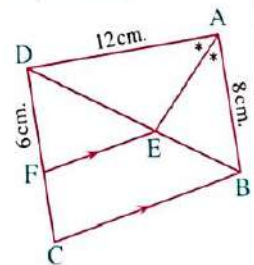
In the opposite figure :

ABCD is a quadrilateral in which : AB = 8 cm.

, AD = 12 cm., \overrightarrow{AE} bisects $\angle A$ and intersects \overline{BD} at E

, $\overrightarrow{EF} \parallel \overline{BC}$ and intersects \overline{DC} at F, if DF = 6 cm.,

then find the length of : \overline{DC}



Important Remarks

1 In the opposite figure :

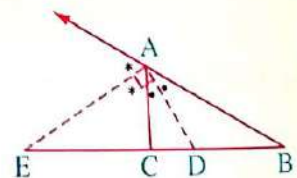
If \overrightarrow{AD} , \overrightarrow{AE} are the bisectors of the angle A and
the exterior angle of $\triangle ABC$ at A respectively

$$\text{, then } \frac{BD}{DC} = \frac{AB}{AC}, \frac{BE}{EC} = \frac{AB}{AC} \quad \therefore \frac{BD}{DC} = \frac{BE}{EC}$$

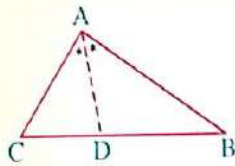
\therefore The base \overline{BC} is divided internally at D, externally at E by the same ratio (AB : AC)

and we notice that : the two bisectors \overrightarrow{AD} and \overrightarrow{AE} are perpendicular.

i.e. $\angle DAE = 90^\circ$



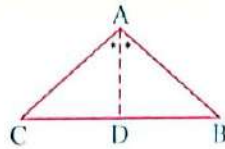
2 If \overrightarrow{AD} bisects $\angle BAC$ and intersects \overline{BC} at D, then D takes one of the following :



If $AB > AC$

, then $BD > DC$

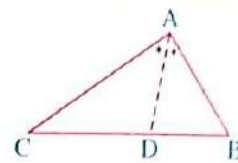
i.e. D is nearer to C than to B



If $AB = AC$

, then $BD = DC$

i.e. D is equidistant from each of B and C

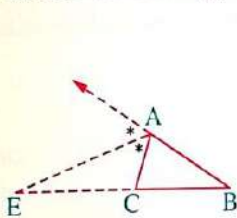


If $AB < AC$

, then $BD < DC$

i.e. D is nearer to B than to C

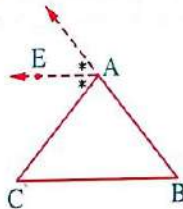
3 If \overrightarrow{AE} bisects the exterior angle of $\triangle ABC$ at A, where $E \notin \overline{BC}$, then E takes one of the following :



If $AB > AC$

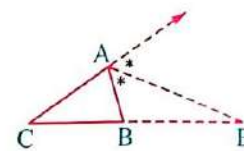
, then $BE > EC$

i.e. $E \in \overline{BC}$



If $AB = AC$

, then $\overrightarrow{AE} \parallel \overline{BC}$



If $AB < AC$

, then $BE < EC$

i.e. $E \in \overline{CB}$

Example 5

ABC is a triangle in which $AB = 8$ cm. , $AC = 6$ cm. , $BC = 7$ cm. , draw \overrightarrow{AD} to bisect $\angle A$ and intersect \overline{BC} at D , draw \overrightarrow{AE} to bisect the exterior angle A and intersect \overline{BC} at E
Find the length of : \overline{DE}

Solution

In $\triangle ABC$:

$\therefore \overrightarrow{AD}$ bisects $\angle A$, \overrightarrow{AE} bisects the exterior angle A

$$\therefore \frac{BD}{DC} = \frac{BE}{CE} = \frac{BA}{AC}$$

$$\therefore \frac{BD}{DC} = \frac{BE}{CE} = \frac{8}{6} = \frac{4}{3} \quad (1)$$

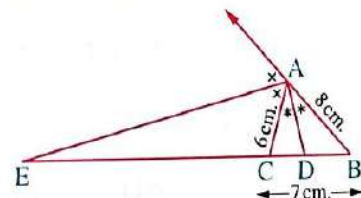
$$\therefore \frac{BD + DC}{DC} = \frac{4 + 3}{3}$$

(from the properties of the proportion)

$$\therefore \frac{BC}{DC} = \frac{7}{3}$$

$$\therefore \frac{7}{DC} = \frac{7}{3}$$

$$\therefore DC = 3 \text{ cm.}$$



From (1) : $\therefore \frac{BE}{EC} = \frac{4}{3}$ $\therefore \frac{BE - EC}{CE} = \frac{4 - 3}{3}$ (from the properties of the proportion)

$\therefore \frac{BC}{CE} = \frac{1}{3}$ $\therefore \frac{7}{CE} = \frac{1}{3}$

$\therefore CE = 21 \text{ cm.}$ $\therefore DE = DC + CE = 3 + 21 = 24 \text{ cm.}$ (The req.)

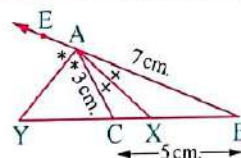
TRY TO SOLVE

In the opposite figure :

\overrightarrow{AX} bisects $\angle BAC$, \overrightarrow{AY} bisects $\angle CAE$

, $AB = 7 \text{ cm.}$, $AC = 3 \text{ cm.}$, $BC = 5 \text{ cm.}$

Find the length of : \overline{XY}

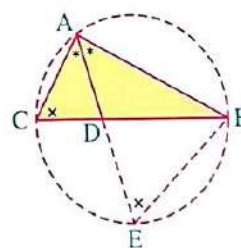


Finding the lengths of the interior and the exterior bisectors of an angle of a triangle

Well known problem

If \overrightarrow{AD} bisects $\angle A$ in $\triangle ABC$ internally and intersects \overline{BC} at D
 , then $AD = \sqrt{AB \times AC - BD \times DC}$

- **Given** ABC is a triangle, \overrightarrow{AD} bisects $\angle BAC$ internally
 , $\overrightarrow{AD} \cap \overline{BC} = \{D\}$
- **R.T.P.** $AD = \sqrt{AB \times AC - BD \times DC}$
- **Const.** Draw a circle passes through the vertices of $\triangle ABC$
 and intersects \overline{AD} at E , draw \overline{BE}



- **Proof** $\therefore m(\angle CAD) = m(\angle EAB)$ (given)
- $m(\angle E) = m(\angle C)$ (inscribed angles subtended by \widehat{AB})
- $\therefore \triangle ACD \sim \triangle AEB$, then $\frac{AC}{AE} = \frac{AD}{AB}$
- $\therefore AD \times AE = AB \times AC$
- $\therefore AD \times (AD + DE) = AB \times AC$
- $\therefore (AD)^2 = AB \times AC - AD \times DE$
- $\therefore (AD)^2 = AB \times AC - BD \times DC$
- $\therefore AD = \sqrt{AB \times AC - BD \times DC}$ (Q.E.D.)

Remember that

$$AD \times DE = BD \times DC$$

Example 6

ABC is a triangle in which : $AB = 15$ cm. , $AC = 9$ cm. , \overrightarrow{AD} bisects $\angle BAC$ and intersects \overline{BC} at D , if $DC = 6$ cm.

Find the length of : \overline{AD}

Solution

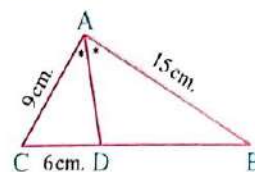
$\therefore \overrightarrow{AD}$ bisects $\angle BAC$

$$\therefore \frac{BD}{DC} = \frac{BA}{CA}$$

$$\therefore \frac{BD}{6} = \frac{15}{9}$$

$$\therefore BD = \frac{15 \times 6}{9} = 10 \text{ cm.}$$

$$\therefore AD = \sqrt{AB \times AC - BD \times DC} = \sqrt{15 \times 9 - 10 \times 6} = \sqrt{75} = 5\sqrt{3} \text{ cm.}$$



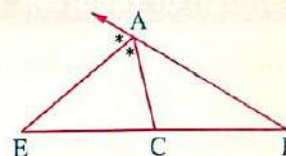
(The req.)

Remark

In the opposite figure :

If \overrightarrow{AE} bisects $\angle BAC$ externally and intersects \overline{BC} at E

, then $AE = \sqrt{BE \times EC - AB \times AC}$



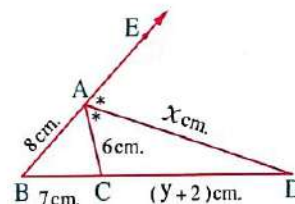
Example 7

In the opposite figure :

ABC is a triangle in which $AB = 8$ cm.

, $BC = 7$ cm. , $AC = 6$ cm. , \overrightarrow{AD} bisects $\angle A$ externally.

Find the value of each of : x , y



Solution

$\therefore \overrightarrow{AD}$ bisects $\angle A$ externally

$$\therefore \frac{BD}{CD} = \frac{BA}{AC} = \frac{8}{6} = \frac{4}{3}$$

$$\therefore \frac{7+y+2}{y+2} = \frac{4}{3}$$

$$\therefore \frac{y+9}{y+2} = \frac{4}{3}$$

$$\therefore 3y + 27 = 4y + 8$$

$$\therefore y = 19$$

$$\therefore DC = 21 \text{ cm.} , BD = 28 \text{ cm.}$$

$$\therefore AD = \sqrt{BD \times CD - BA \times AC} = \sqrt{28 \times 21 - 8 \times 6} = \sqrt{540} = 6\sqrt{15} \text{ cm.}$$

$$\therefore x = 6\sqrt{15}$$

(The req.)

TRY TO SOLVE

ABC is a triangle in which : $AB = 27$ cm. , $AC = 15$ cm. , draw \overrightarrow{AD} to bisect $\angle A$ and intersect \overline{BC} at D , if $BD = 18$ cm.

Find the length of : \overline{AD}

Lesson

4



Follow : Angle bisector and proportional parts

Converse of theorem 3

In the opposite two figures :

- If $D \in \overline{BC}$ (Fig. 1)

such that : $\frac{BD}{DC} = \frac{BA}{AC}$

, then \overrightarrow{AD} bisects $\angle BAC$

- If $D \in \overline{BC}$, $D \notin \overline{BC}$ (Fig. 2)

such that : $\frac{BD}{DC} = \frac{BA}{AC}$

, then \overrightarrow{AD} bisects the exterior angle of $\triangle ABC$ at A

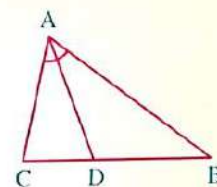


Fig. (1)

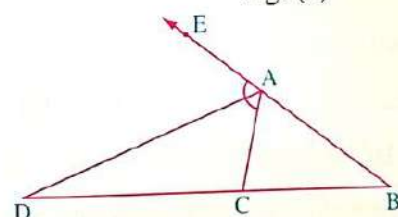


Fig. (2)

Example 1

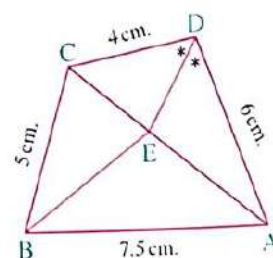
In the opposite figure :

ABCD is a quadrilateral in which $AB = 7.5$ cm.

, $BC = 5$ cm. , $CD = 4$ cm. , $AD = 6$ cm.

, \overrightarrow{DE} bisects $\angle ADC$ and intersects \overline{AC} at E

Prove that : \overrightarrow{BE} bisects $\angle ABC$



Solution

In $\triangle ACD$: $\because \overrightarrow{DE}$ bisects $\angle ADC$

, $\therefore \frac{AE}{EC} = \frac{AD}{DC} = \frac{6}{4} = \frac{3}{2}$

\therefore In $\triangle ABC$: \overrightarrow{BE} bisects $\angle ABC$

$\therefore \frac{AE}{EC} = \frac{AD}{DC} = \frac{6}{4} = \frac{3}{2}$

$\therefore \frac{AE}{EC} = \frac{AB}{BC}$

(Q.E.D.)

Example 2

ABC is an isosceles triangle in which $AB = AC$, $D \in \overline{BC}$, where $BC = CD$, draw the bisector of the angle ABC to intersect \overline{AC} at E, draw $\overline{EF} \parallel \overline{BC}$ and intersects \overline{AD} at F

Prove that : \overline{CF} bisects $\angle ACD$

Solution

In $\triangle ABC$: $\because \overline{BE}$ bisects $\angle ABC$

$$\therefore \frac{AE}{EC} = \frac{AB}{BC}, \text{ but } AB = AC, BC = CD \quad (\text{given})$$

$$\therefore \frac{AE}{EC} = \frac{AC}{CD} \quad (1)$$

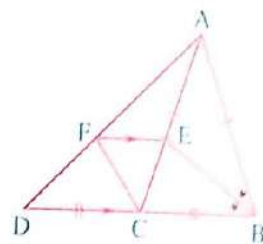
In $\triangle ACD$:

$$\because \overline{EF} \parallel \overline{CD} \quad \therefore \frac{AE}{EC} = \frac{AF}{FD} \quad (2)$$

$$\text{From (1), (2) : } \therefore \frac{AF}{FD} = \frac{AC}{CD}$$

\therefore In $\triangle ACD$: \overline{CF} bisects $\angle ACD$

(Q.E.D.)

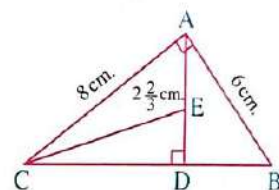

Example 3

In the opposite figure :

ABC is a right-angled triangle at A, $\overline{AD} \perp \overline{BC}$

, $AB = 6$ cm. , $AC = 8$ cm. , $AE = 2\frac{2}{3}$ cm.

Prove that : \overline{CE} bisects $\angle ACD$


Solution

$\because \triangle ABC$ is right-angled at A

$$\therefore (BC)^2 = (AB)^2 + (AC)^2 = 36 + 64 = 100$$

$\therefore BC = 10$ cm.

$\because \overline{AD} \perp \overline{BC}$

$\therefore \triangle DAC \sim \triangle ABC$

$$\therefore \frac{DC}{AC} = \frac{AC}{BC}$$

$$\therefore \frac{DC}{8} = \frac{8}{10} \quad \therefore DC = 6.4 \text{ cm.}$$

$\because \triangle DBA \sim \triangle ABC$

$$\therefore \frac{AB}{CB} = \frac{AD}{CA}$$

$$\therefore \frac{6}{10} = \frac{AD}{8}$$

$$\therefore AD = 4.8 \text{ cm.} \quad \therefore DE = 4.8 - 2\frac{2}{3} = 2\frac{2}{15} \text{ cm.}$$

$$\therefore \frac{AC}{CD} = \frac{8}{6.4} = \frac{5}{4}, \quad \frac{AE}{ED} = \frac{2\frac{2}{3}}{2\frac{2}{15}} = \frac{5}{4}$$

$$\therefore \frac{AC}{CD} = \frac{AE}{ED}$$

$\therefore \overline{CE}$ bisects $\angle ACD$

(Q.E.D.)

TRY TO SOLVE

ABCD is a quadrilateral in which $AB = 20$ cm. , $AD = 6$ cm. , $DC = 9$ cm. , $E \in \overline{AB}$ such that $AE = 8$ cm. , draw $\overline{EX} \parallel \overline{BC}$ to intersect \overline{AC} at X

Prove that : \overline{DX} bisects $\angle ADC$

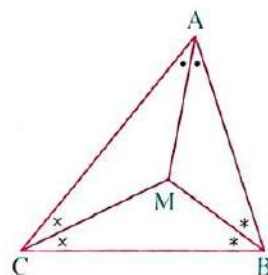
Fact

The bisectors of angles of a triangle are concurrent.

In the opposite figure :

\overline{AM} , \overline{BM} and \overline{CM} are concurrent

at the point M



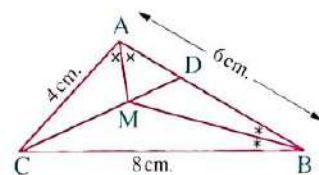
Example 4

In the opposite figure :

ABC is a triangle in which $AB = 6$ cm. , $AC = 4$ cm.

, $BC = 8$ cm. , \overline{BM} bisects $\angle ABC$, \overline{AM} bisects $\angle BAC$

Find the length of : \overline{AD}



Solution

$\therefore \overline{AM}$ bisects $\angle BAC$, \overline{BM} bisects $\angle ABC$

$\therefore M$ is the point of concurrence of the bisectors of angles of $\triangle ABC$

$\therefore \overline{CM}$ bisects $\angle ACB$

\therefore In $\triangle ABC$: $\frac{AD}{DB} = \frac{AC}{CB} = \frac{4}{8} = \frac{1}{2}$

$\therefore \frac{AD}{6 - AD} = \frac{1}{2}$

$\therefore 2AD = 6 - AD$

$\therefore 3AD = 6$

$\therefore AD = 2$ cm.

(The req.)

TRY TO SOLVE

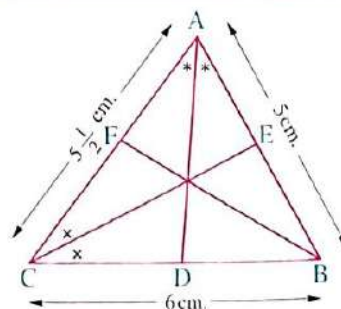
In the opposite figure :

ABC is a triangle in which $AB = 5$ cm.

, $AC = 5\frac{1}{2}$ cm. , $BC = 6$ cm.

, \overline{AD} bisects $\angle BAC$, \overline{CE} bisects $\angle ACB$

Find the length of : \overline{AF}



Lesson

5

Applications of proportionality in the circle

Power of a point with respect to a circle

Definition

Power of the point A with respect to the circle M in which the length of its radius is r , is the real number $P_M(A)$ where $P_M(A) = (AM)^2 - r^2$

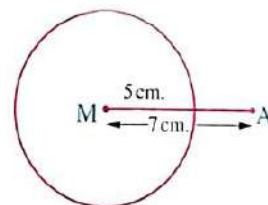
For example :

In the opposite figure :

If A is a point outside the circle M whose radius length equals 5 cm.

, where $MA = 7$ cm.

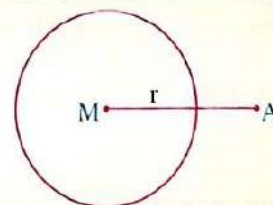
, then $P_M(A) = 7^2 - 5^2 = 24$



Note 1

We can expect the position of point A with respect to the circle M if :

- $P_M(A) > 0$, then A lies outside the circle.
- $P_M(A) = 0$, then A lies on the circle.
- $P_M(A) < 0$, then A lies inside the circle.



Example 1

If M is a circle of diameter length 12 cm. , A is a point lies on its plane , determine the position of point A with respect to the circle M in each of the following cases , then calculate its distance from the centre of the circle :

1 $P_M(A) = 13$

2 $P_M(A) = \text{Zero}$

3 $P_M(A) = -11$

Solution

\therefore Length of circle diameter = 12 cm. $\therefore r = 6$ cm.

1 $\therefore P_M(A) = 13 > 0$

\therefore A lies outside the circle

$\therefore P_M(A) = (MA)^2 - r^2$

$\therefore 13 = (MA)^2 - 36$

$\therefore MA = 7$ cm.

2 $\therefore P_M(A) = \text{Zero}$

\therefore A lies on the circle

$\therefore MA = 6$ cm.

3 $\therefore P_M(A) = -11 < 0$

\therefore A lies inside the circle

$\therefore P_M(A) = (MA)^2 - r^2$

$\therefore -11 = (MA)^2 - 36$

$\therefore MA = 5$ cm.

TRY TO SOLVE

Determine the position of each of the points A, B and C with respect to the circle M whose radius length is 5 cm. if :

1 $P_M(A) = 11$

2 $P_M(B) = \text{Zero}$

3 $P_M(C) = -16$

Then calculate the distance of each point from the circle centre M

Note 2

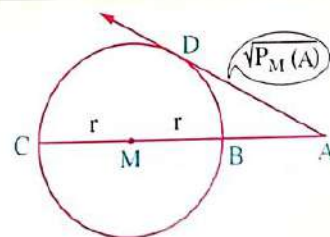
If the point A lies outside the circle M

\therefore then $P_M(A) = (AM)^2 - r^2$

$= (AM - r)(AM + r)$

$= AB \times AC = (AD)^2$

\therefore length of the tangent drawn from A to circle M $= \sqrt{P_M(A)}$

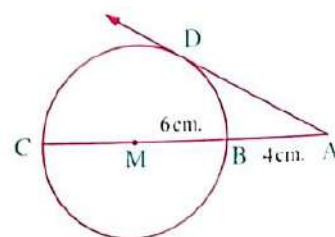


For example : In the opposite figure :

If point A lies outside the circle M whose radius length is 6 cm. \overline{AD} is a tangent to the circle at D

If $AB = 4$ cm. \therefore we can find $P_M(A)$

with one of the following methods :



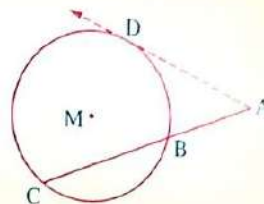
- Using the definition : $P_M(A) = (AM)^2 - r^2 = (10)^2 - (6)^2 = 64$
- Using the previous note : $P_M(A) = AB \times AC = 4 \times 16 = 64$

From the previous , we can get : AD where $AD = \sqrt{P_M(A)} = \sqrt{64} = 8$ cm.

Notice that

In the opposite figure :

If point A lies outside the circle , \overline{AC} intersects the circle at B , C
 , then $P_M(A) = AB \times AC$



And this can be concluded from the previous note , where :

$$P_M(A) = (AD)^2 \quad , \text{ where } \overline{AD} \text{ is a tangent to the circle M at D}$$

$$, \therefore (AD)^2 = AB \times AC \quad \therefore P_M(A) = AB \times AC$$

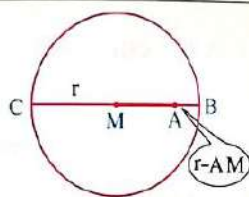
Note 3

If point A lies inside the circle M , then :

$$P_M(A) = (AM)^2 - r^2$$

$$= (AM - r)(AM + r)$$

$$= -(r - AM)(AM + r) = -AB \times AC$$

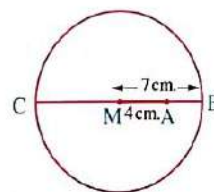


For example : In the opposite figure :

If point A lies inside the circle M whose radius length is 7 cm.

and lies at a distance of 4 cm. from the circle centre

, then $P_M(A) = -AB \times AC = -3 \times 11 = -33$

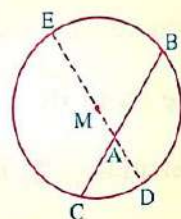


Notice that

In the opposite figure :

If \overline{BC} is a chord in the circle M , $A \in \overline{BC}$

, then $P_M(A) = -AB \times AC$



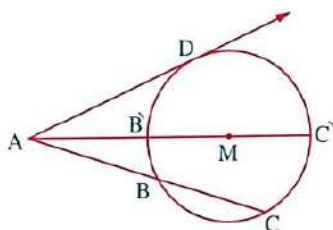
And this could be concluded from the previous note as follows :

$$P_M(A) = -AD \times AE \quad (\text{where } \overline{DE} \text{ is a diameter})$$

$$\therefore AD \times AE = AB \times AC \quad \therefore P_M(A) = -AB \times AC$$

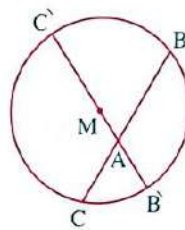
Summary of the previous as follows :

If A lies outside circle M , then :



$$P_M(A) = AB \times AC = \overline{AB} \times \overline{AC} = (\overline{AD})^2$$

If A lies inside circle M , then :



$$P_M(A) = -AB \times AC = -\overline{AB} \times \overline{AC}$$

Example 2

A circle of centre M and its radius length is 3 cm. , A is a point at a distance of 7 cm. from its centre , from A a straight line is drawn to intersect the circle at C , D , where $C \in \overline{AD}$, if $CA = 5$ cm. , calculate the length of the chord \overline{CD}

Solution

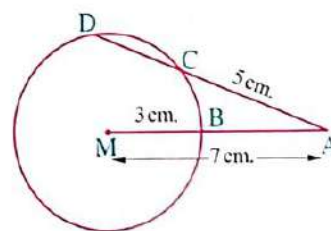
$$\therefore P_M(A) = (AM)^2 - r^2 = 49 - 9 = 40$$

$$\therefore P_M(A) = AC \times AD$$

$$\therefore 40 = 5 \times AD$$

$$\therefore AD = 8 \text{ cm.}$$

$$\therefore CD = AD - AC = 8 - 5 = 3 \text{ cm.}$$



(The req.)

Example 3

A circle M of radius length 7 cm. , A is a point at a distance of 5 cm. from its centre.

The chord \overline{BC} passes through point A , where $AB = 3 AC$

Calculate : 1 The length of the chord \overline{BC}

2 The distance between \overline{BC} and the centre of the circle.

Solution

$$\therefore P_M(A) = (AM)^2 - r^2 = 25 - 49 = -24$$

$$\therefore P_M(A) = -AB \times AC$$

$$\therefore -24 = -AB \times AC$$

$$\therefore 24 = AB \times AC$$

$$\therefore AB = 3 AC$$

$$\therefore 24 = 3 AC \times AC$$

$$\therefore 8 = (AC)^2$$

$$\therefore AC = \sqrt{8} = 2\sqrt{2} \text{ cm.}$$

$$\therefore AB = 3 AC$$

$$\therefore AB = 6\sqrt{2} \text{ cm.}$$

$$\therefore BC = AC + AB = 8\sqrt{2} \text{ cm.}$$

(First req.)

, let the distance between the chord \overline{BC} and the centre of the circle be MD

, where $\overline{MD} \perp \overline{BC}$

$$\therefore \overline{MD} \perp \overline{BC}$$

$\therefore D$ is the midpoint of \overline{BC}

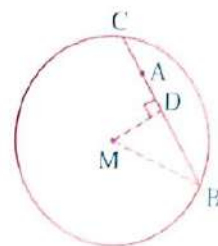
$$\therefore P_M(D) = (DM)^2 - r^2 = -BD \times DC$$

$$\therefore (DM)^2 - 49 = -4\sqrt{2} \times 4\sqrt{2}$$

$$\therefore (DM)^2 = 17$$

$$\therefore DM = \sqrt{17} \approx 4.1 \text{ cm}$$

(Second req.)



TRY TO SOLVE

The circle M has radius length 20 cm. , A is a point at a distance 16 cm.

from the centre of the circle , the chord \overline{BC} is drawn where $A \in \overline{BC}$, $AB = 2 AC$

Calculate : 1 The length of the chord \overline{BC}

2 The distance between the chord \overline{BC} and the centre of the circle.

Important Note

The set of points which have the same power with respect to two distinct circles is called the principle axis of the two circles.

If $P_M(A) = P_N(A)$, then A lies on the principle axis of the two circles M and N

For example :

If $P_M(A) = P_N(A)$, $P_M(B) = P_N(B)$

, then \overleftrightarrow{AB} is the principle axis of the two circles M and N

Example 4

Two circles M and N are intersecting at A and B , $C \in \overleftrightarrow{BA}$, $C \notin \overleftrightarrow{BA}$, draw \overleftrightarrow{CD} to intersect the circle M at D and E , where $CD = 9$ cm. , $DE = 7$ cm. , draw \overleftrightarrow{CF} to touch the circle N at F

1 Prove that : C lies on the principle axis of the two circles M and N

2 If $AB = 10$ cm. , find the length of each of : \overline{AC} , \overline{CF}

Solution

\therefore A lies on the circle M , A lies on the circle N

$\therefore P_M(A) = P_N(A) = \text{zero}$,

Similarly : $P_M(B) = P_N(B) = \text{zero}$

$\therefore \overleftrightarrow{AB}$ is the principle axis for the two circles M and N

, $\therefore C \in \overleftrightarrow{AB}$

\therefore C lies on the principle axis of the two circles M and N

, $\therefore P_M(C) = CD \times CE = 9 \times 16 = 144$

, $P_M(C) = CA \times CB$

$\therefore 144 = (CA)^2 + 10 CA$

$\therefore (CA - 8)(CA + 18) = 0$

, \therefore C lies on the principle axis of the two circles M and N

$\therefore P_N(C) = P_M(C)$, $P_N(C) = (CF)^2$

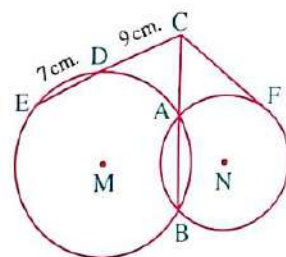
$\therefore (CF)^2 = 144$

$\therefore 144 = CA(CA + 10)$

$\therefore (CA)^2 + 10 CA - 144 = 0$

$\therefore CA = 8$ cm.

$\therefore CF = 12$ cm



(First req.)

(Second req.)

Secant, tangent and measures of angles

Remember that

- 1** The measure of an angle formed by two chords that intersect inside a circle is equal to half the sum of the measures of the intercepted arcs.

In the opposite figure :

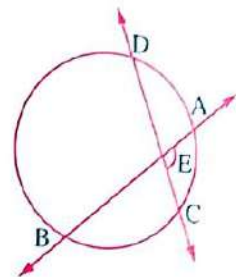
\overleftrightarrow{AB} , \overleftrightarrow{CD} are two secants to the circle, where

$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{E\}$, then

$$m(\angle AEC) = \frac{1}{2}[m(\widehat{AC}) + m(\widehat{BD})]$$

For example If $m(\widehat{AC}) = 50^\circ$, $m(\widehat{BD}) = 170^\circ$

$$\therefore m(\angle AEC) = \frac{1}{2}(50^\circ + 170^\circ) = 110^\circ$$



- 2** The measure of an angle formed by two secants drawn from a point outside a circle is equal to half the positive difference of the measures of the intercepted arcs.

In the opposite figure :

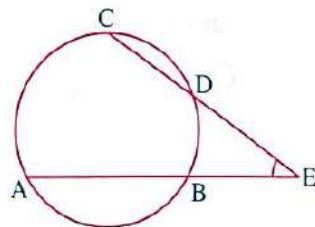
\overleftrightarrow{AB} , \overleftrightarrow{CD} are two secants to the circle, where

$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{E\}$, then

$$m(\angle E) = \frac{1}{2}[m(\widehat{AC}) - m(\widehat{BD})]$$

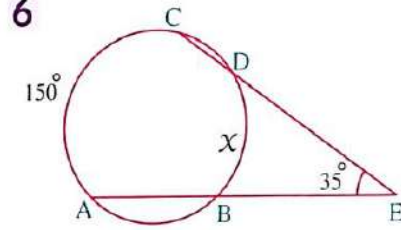
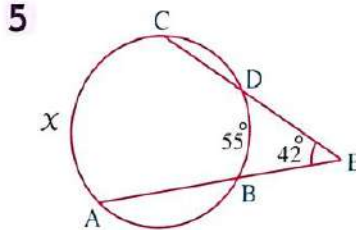
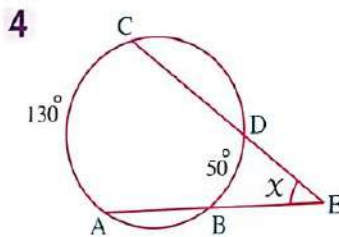
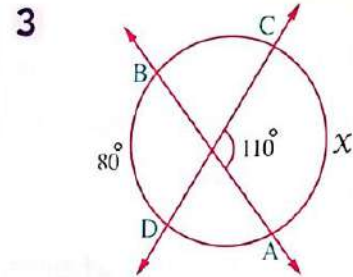
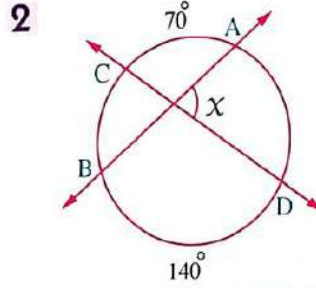
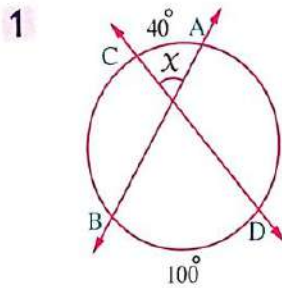
For example If $m(\widehat{AC}) = 120^\circ$, $m(\widehat{BD}) = 50^\circ$

$$\therefore m(\angle E) = \frac{1}{2}[120^\circ - 50^\circ] = 35^\circ$$



Example 5

In each of the following figures, find the value of x :



Solution

1 $x = \frac{1}{2} (40^\circ + 100^\circ) = 70^\circ$

2 \therefore measure of circle = 360° , $m(\widehat{AC}) + m(\widehat{DB}) = 70^\circ + 140^\circ = 210^\circ$

$\therefore m(\widehat{AD}) + m(\widehat{BC}) = 360^\circ - 210^\circ = 150^\circ$

$\therefore x = \frac{1}{2} \times 150^\circ = 75^\circ$

3 $\therefore \frac{1}{2} (x + 80^\circ) = 110^\circ$

$\therefore x + 80^\circ = 220^\circ$

$\therefore x = 140^\circ$

4 $x = \frac{1}{2} (130^\circ - 50^\circ) = 40^\circ$

5 $\therefore \frac{1}{2} (x - 55^\circ) = 42^\circ$

$\therefore x - 55^\circ = 84^\circ$

$\therefore x = 139^\circ$

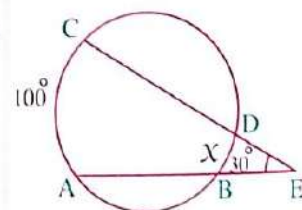
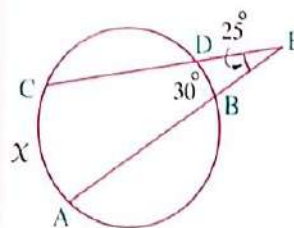
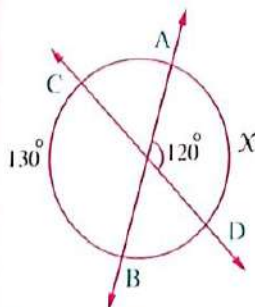
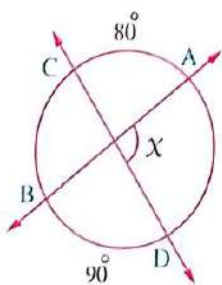
6 $\therefore \frac{1}{2} (150^\circ - x) = 35^\circ$

$\therefore 150^\circ - x = 70^\circ$

$\therefore x = 80^\circ$

TRY TO SOLVE

Find the value of x in each of the following:



Well known problem

The measure of an angle formed by a secant and a tangent or two tangents drawn from a point outside a circle is equal to half the positive difference of the measures of the intercepted arcs.

First case Intersection of a secant and a tangent to a circle

► **Given** \overrightarrow{AB} is a tangent to the circle M at B, $\overrightarrow{AD} \cap$ the circle M = {C, D}

► **R.T.P.** $m(\angle A) = \frac{1}{2} [m(\widehat{BD}) - m(\widehat{BC})]$

► **Const.** Draw \overline{BC} , \overline{BD}

► **Proof** $\therefore \angle BCD$ is an exterior angle of $\triangle ABC$

$$\therefore m(\angle BCD) = m(\angle A) + m(\angle ABC)$$

$$\therefore m(\angle A) = m(\angle BCD) - m(\angle ABC)$$

$\therefore \angle BCD$ is an inscribed angle.

$$\therefore m(\angle BCD) = \frac{1}{2} m(\widehat{BD})$$

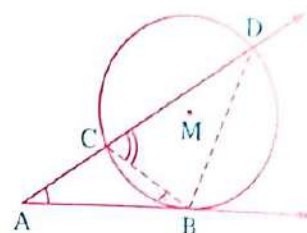
$\therefore \angle ABC$ is a tangency angle.

$$\therefore m(\angle ABC) = \frac{1}{2} m(\widehat{BC})$$

$$\therefore m(\angle A) = \frac{1}{2} m(\widehat{BD}) - \frac{1}{2} m(\widehat{BC})$$

$$= \frac{1}{2} [m(\widehat{BD}) - m(\widehat{BC})]$$

(Q.E.D.)



Second case Intersection of two tangents to a circle

► **Given** \overrightarrow{AB} , \overrightarrow{AC} are two tangents to the circle M at B and C

► **R.T.P.** $m(\angle A) = \frac{1}{2} [m(\widehat{BXC}) - m(\widehat{BC})]$

► **Const.** Draw \overline{BC}

► **Proof** $\therefore \angle BCD$ is an exterior angle of $\triangle ABC$

$$\therefore m(\angle BCD) = m(\angle A) + m(\angle B)$$

$$\therefore m(\angle A) = m(\angle BCD) - m(\angle B)$$

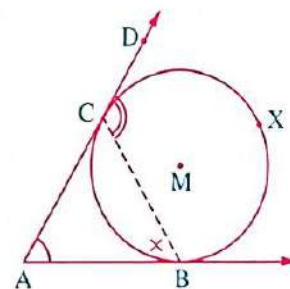
$$\therefore \angle BCD \text{ is a tangency angle.} \quad \therefore m(\angle BCD) = \frac{1}{2} m(\widehat{BXC})$$

$$\therefore \angle B \text{ is a tangency angle.} \quad \therefore m(\angle B) = \frac{1}{2} m(\widehat{BC})$$

$$\therefore m(\angle A) = \frac{1}{2} m(\widehat{BXC}) - \frac{1}{2} m(\widehat{BC})$$

$$= \frac{1}{2} [m(\widehat{BXC}) - m(\widehat{BC})]$$

(Q.E.D.)



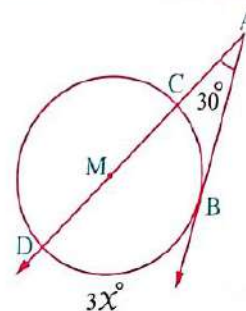
Example 6

In the opposite figure :

If \overline{AB} is a tangent to the circle M at B , $m(\angle A) = 30^\circ$

, \overline{AD} is a secant to the circle at C and D , $m(\widehat{BD}) = 3x^\circ$

Find the values of : x



Solution

$\therefore \overline{AB}$ is a tangent to the circle M , \overline{AD} is a secant to it.

$$\therefore m(\angle A) = \frac{1}{2} [m(\widehat{BD}) - m(\widehat{BC})] \quad \therefore \frac{1}{2} [m(\widehat{BD}) - m(\widehat{BC})] = 30^\circ$$

$$\therefore m(\widehat{BD}) - m(\widehat{BC}) = 60^\circ \quad (1)$$

$$\therefore \overline{CD} \text{ is a diameter in the circle M} \quad \therefore m(\widehat{BD}) + m(\widehat{BC}) = 180^\circ \quad (2)$$

Adding (1) , (2) we get that : $2m(\widehat{BD}) = 240^\circ$

$$\therefore m(\widehat{BD}) = 120^\circ$$

$$\therefore m(\widehat{BD}) = 3x^\circ \quad \therefore 3x^\circ = 120^\circ \quad \therefore x = 40^\circ \quad (\text{The req.})$$

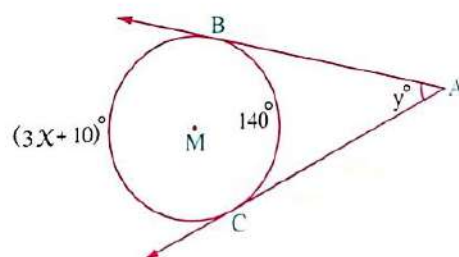
Example 7

In the opposite figure :

If \overline{AB} and \overline{AC} are two tangents to the circle M at B , C respectively , $m(\angle A) = y^\circ$

, $m(\widehat{BC}) \text{ minor} = 140^\circ$, $m(\widehat{BC}) \text{ major} = (3x + 10)^\circ$

Find the values of : x and y



Solution

\therefore The measure of the circle = 360°

$$\therefore m(\widehat{BC}) \text{ minor} + m(\widehat{BC}) \text{ major} = 360^\circ$$

$$\therefore 140^\circ + (3x + 10)^\circ = 360^\circ \quad \therefore 3x^\circ + 150^\circ = 360^\circ$$

$$\therefore 3x^\circ = 210^\circ \quad \therefore x = 70^\circ$$

$$\therefore m(\widehat{BC}) \text{ major} = (3 \times 70^\circ + 10^\circ) = 220^\circ$$

$\therefore \overline{AB}$ and \overline{AC} are two tangents to circle M

$$\therefore m(\angle A) = \frac{1}{2} [m(\widehat{BC}) \text{ major} - m(\widehat{BC}) \text{ minor}]$$

$$\therefore y = \frac{1}{2} [220^\circ - 140^\circ] = 40^\circ \quad (\text{The req.})$$

TRY TO SOLVE

Using the givens in the figure, find the value of the symbol used in measurement :

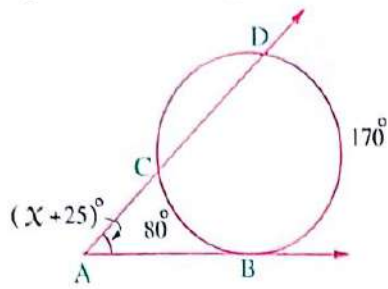


Fig. (1)

$$X = \dots\dots\dots^\circ$$

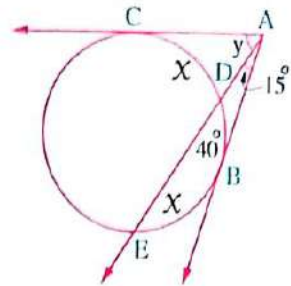


Fig. (2)

$$X = \dots\dots\dots^\circ, y = \dots\dots\dots^\circ$$



Mathematics

By a group of supervisors

Exercises

This book includes new problems
measuring higher levels of thinking



FIRST TERM

1

SEC.
2020



EL-MONASSER

FREE PART 1

Unit 1

Algebra, Relations and Functions

Unit Exercises

Pre-requirements on unit one.

Exercise 1 : An introduction in complex numbers.

Exercise 2 : Determining the types of roots of a quadratic equation.

Exercise 3 : Relation between the two roots of the second degree equation and the coefficients of its terms.

Exercise 4 : Forming the quadratic equation whose two roots are known.

Exercise 5 : Sign of a function.

Exercise 6 : Quadratic inequalities in one variable.

At the end of the unit :

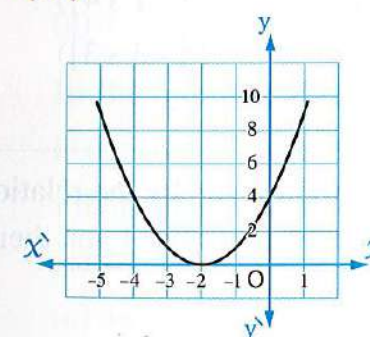
- Life applications on unit one.

Pre-requirements on unit one

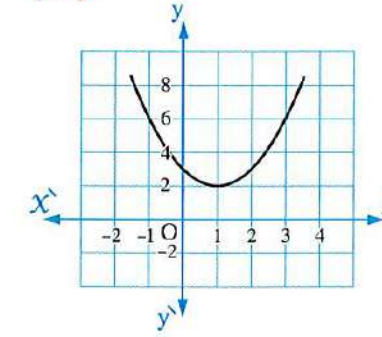
From the school book

- 1** Each of the following graphs illustrates a quadratic function. Find the solution set in \mathbb{R} of the equation $f(x) = 0$ in each figure :

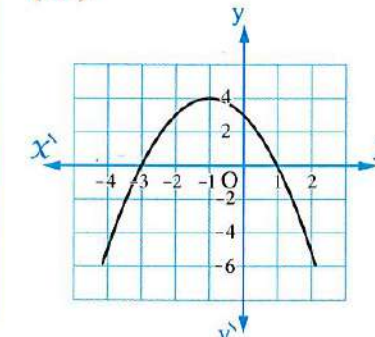
(1)



(2)



(3)



- 2** Choose the correct answer from those given :

- (1) If the curve of the quadratic function f cuts the x -axis at the two points $(2, 0)$, $(-3, 0)$, then the solution set of $f(x) = 0$ in \mathbb{R} is
- (a) $\{2, 0\}$ (b) $\{-3, 0\}$ (c) $\{-3, 2\}$ (d) $\{(2, -3)\}$
- (2) The solution set of the equation : $x^2 - x = 0$ in \mathbb{R} is
- (a) $\{1, -1\}$ (b) $\{0\}$ (c) $\{0, 1\}$ (d) \emptyset
- (3) If $f(x) = x^2 + bx + c$, $x = 2$ is a root of the equation : $f(x) = 0$, then $f(2) = \dots\dots\dots$
- (a) 2 (b) -2 (c) 4 (d) zero
- (4) If $x = 3$ is a root of the equation : $x^2 + mx = 3$, then $m = \dots\dots\dots$
- (a) -1 (b) -2 (c) 2 (d) 1

- (5) If one of the roots of the equation : $x^2 - 16 = 0$ is 4 , then the other root is
- (a) -4 (b) 4 (c) 8 (d) zero

3 Find algebraically the solution set of each of the following equations in \mathbb{R} :

- (1) $x^2 - 1 = 0$ (2) $x^2 + 9 = 0$
 (3) $x^2 + 3x = 0$ (4) $x^2 - 6x + 9 = 0$

4 Find in \mathbb{R} the solution set of each of the following equations by using the general formula approximating the result to the nearest tenth :

- (1) $x^2 - 6x + 1 = 0$ (2) $x^2 + 3x + 5 = 0$
 (3) $2x^2 + 3x - 4 = 0$ (4) $3x^2 - 65 = 0$
 (5) $x - \frac{5}{x} = 3$ (6) $\frac{3}{x-2} + \frac{2}{x+2} = 2$

5 Find in \mathbb{R} the solution set of each of the following equations algebraically , then check the answer graphically :

- (1) $x^2 - 2x - 4 = 0$ (Draw graphically in the interval $[-2, 4]$)
 (2) $3x - x^2 + 2 = 0$ (Draw graphically in the interval $[-1, 4]$)
 (3) $x^2 + 3 = 0$ (Draw graphically in the interval $[-3, 3]$)
 (4) $-2x^2 - 4x + 1 = 0$

6 If the sum of the whole consecutive numbers $(1 + 2 + 3 + \dots + n)$ is given by the relation $S = \frac{n}{2}(1 + n)$, how many whole consecutive numbers starting from the number 1 and their sum equals :

- (1) 78 (2) 171 (3) 253 (4) 465

7 Choose the correct answer from the given ones :

- (1) The necessary condition which makes the equation $ax^2 + bx + c = 0$ quadratic is
- (a) $a > 0$ (b) $a < 0$ (c) $a \neq 0$ (d) $a \neq 0, b \neq 0$
- (2) If $(y - 4)^2 = 36$, $y < 0$, then $y + 4 = \dots\dots\dots$
- (a) -2 (b) 2 (c) 10 (d) 14
- (3) If $x = 4$ is one of the roots of the equation : $x^2 + mx = 4$, then
- (a) $m = -3$ (b) m is an even number
 (c) $(1 - m)$ is a perfect square (d) (a) , (c) are both right
- (4) The common root of the two quadratic equations : $x^2 - 3x + 2 = 0$ and $2x^2 - 5x + 2 = 0$ is
- (a) $x = 2$ (b) $x = 1$ (c) $x = -2$ (d) $x = \frac{1}{2}$

(5) If the curve : $y = x(a - x)$, which of the following statements could be right ?

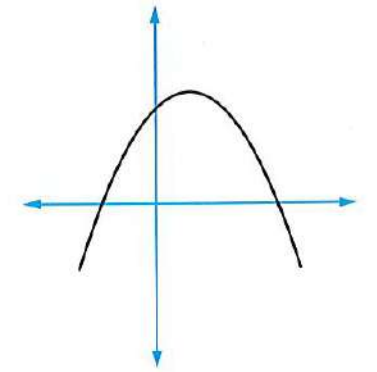
- ① The curve intersects the x -axis at two points $(0, 0)$, $(a, 0)$
 ② The curve vertex is $(\frac{a}{2}, \frac{a^2}{4})$
 ③ The axis of symmetry of the curve is : $x = a$
- (a) ① , ② only (b) ① , ③ only
 (c) ② , ③ only (d) All the previous

(6) The opposite figure represents the curve

$$y = ax^2 + bx + c$$

which of the following is true ?

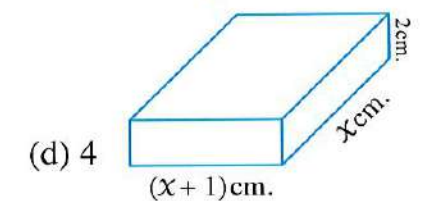
- (a) $a > 0, c > 0$ (b) $a > 0, c < 0$
 (c) $a < 0, c > 0$ (d) $a < 0, c < 0$



(7) In the opposite figure :

If the volume of the cuboid = 40 cm^3 , then $x = \dots\dots\dots \text{ cm}$.

- (a) 7 (b) 6 (c) 5

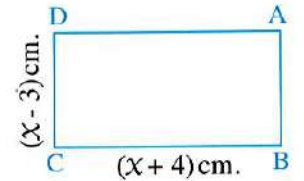


(8) In the opposite figure :

If the area of the rectangle = 78 cm^2

, then the perimeter of the rectangle = $\dots\dots\dots \text{ cm}$.

- (a) 78 (b) 58
 (c) 38 (d) 19



(9) A rectangle piece of land whose dimensions 6 , 9 metres. It's needed to double its area by increasing each dimension by the same length , then the increased length = $\dots\dots\dots \text{ m}$.

- (a) 3 (b) 5 (c) 7 (d) 9

8 Find the value of a which makes $x = 2$ is one of the roots of the equation :

$$x^2 - 2ax + 2(a^2 - 6) = 0$$

$$\ll 1 + \sqrt{5} \text{ or } 1 - \sqrt{5} \gg$$

9 If $f(x) = ax^2 + bx + c$, $f(0) = -3$

, find the value of each of a , b and c if the roots of the equation $f(x) = 0$ are 3 and $-\frac{1}{2}$

$$\ll 2, -5, -3 \gg$$

An introduction
in complex numbers

Test

yourself

From the school book

1 Put each of the following in the simplest form :

(1) i^{24}

(2) i^{26}

(3) i^{43}

(4) i^{-43}

(5) i^{-30}

(6) i^{-100}

(7) $\frac{1}{i^4}$

(8) $\frac{1}{i^{15}}$

(9) $\frac{1}{i^{211}}$

2 If n is an integer, write each of the following in the simplest form :

(1) i^{8n}

(2) i^{4n+42}

(3) i^{12n+3}

(4) i^{8n-3}

3 Simplify each of the following :

(1) $\sqrt{-15}$

(2) $\sqrt{2} \times \sqrt{-8}$

(3) $\sqrt{-18} \times \sqrt{-12}$

(4) $3i(-2i)$

(5) $(-4i)(-6i)$

(6) $(-2i)^3(-3i)^2$

4 Find the result of each of the following in the simplest form :

(1) $(3+2i) + (2-5i)$

(2) $(12-5i^{17}) - (7-\sqrt{-81})$

(3) $2 - (1-2i) + (4-5i) - (1-3i)$

(4) $(2+\sqrt{-9})(3-4i)$

(5) $(4-3i)(4+3i)$

(6) $(2-5i)^2$

(7) $(3-2i)^2 + (3+2i)$

(8) $(1+i)^4$

(9) $(1+\sqrt{-1})^4 - (1-\sqrt{-1})^4$

(10) $(1-i)^{10}$

(11) $(1+2i^2)(2+3i^5+4i^6)$

5 Put each of the following in the form $(a+bi)$ where a and b are real numbers :

(1) $\frac{4-5i}{7i}$

(2) $\frac{26}{3-2i}$

(3) $\frac{2-3i}{3+i}$

(4) $\frac{3+4i}{5-2i}$

(5) $\frac{(3+2i)(2-i)}{3+i}$

(6) $\frac{(3+i)(3-i)}{3-4i}$

(7) $\frac{1}{(1+2i)^2}$

(8) $\frac{1+i+2i^2+2i^3}{1-5i+3i^2-3i^3}$

(9) $\frac{2\sqrt{3}+\sqrt{-8}}{\sqrt{3}-\sqrt{-18}}$

6 Solve each of the following equations in the set of complex numbers :

(1) $3x^2 + 12 = 0$

(2) $4x^2 + 100 = 75$

(3) $x^2 - 4x + 5 = 0$

(4) $2x^2 + 6x + 5 = 0$

7 Find the values of x and y that satisfy each of the following equations :

(1) $x + yi = (3+2i) + (2-i)$

(2) $x + yi = (2-3i)^2$

(3) $3x - 2yi = (5-2i)^2$

(4) $(2x-3) + (3y+1)i = 7+10i$

(5) $(2x-y) + (x-2y)i = 5+i$

(6) $3x + xi - 2y + yi = 5$

(7) $x^2 - y^2 + (x+y)i = 4i$

8 Find the values of x and y that satisfy each of the following equations :

(1) $\frac{10}{2+i} = x + yi$

(2) $\frac{6-4i}{1-i} = x + yi$

(3) $\frac{(2+i)(2-i)}{3+4i} = x + yi$

9 If $x = \frac{13}{5-i}$, $y = \frac{3+2i}{1+i}$, prove that : x and y are two conjugate numbers.**10** If $a+bi = \frac{2+i}{2-i}$, prove that : $a^2 + b^2 = 1$ **11** Choose the correct answer from those given :(1) The conjugate of the number $(3i-4)$ is

(a) $3i+4$

(b) $-3i-4$

(c) $-3i+4$

(d) $3i-4$

(2) The additive inverse of the complex number $(4-7i)$ is

(a) $4+7i$

(b) $-4+7i$

(c) $-4-7i$

(d) $4-7i$

(3) $1+i+i^2+i^3+i^4 = \dots$

(a) $4i+1$

(b) -1

(c) 1

(d) 5

(4) If $12+3ai = 4b-27i$, then $a+b = \dots$

(a) -9

(b) 12

(c) -6

(d) 6

(5) The conjugate of the number (-8) is

(a) $8i$

(b) $-8i$

(c) -8

(d) 8

(6) Which of the following is an imaginary number?

- (a) π (b) $\sqrt{5}$ (c) $\sqrt{-5}$ (d) i^2

(7) $5i^7 + 4i^{-1} = \dots\dots\dots$

- (a) $9i$ (b) $-9i$ (c) i (d) $-i$

(8) If $(1 + i^4)(1 - i^7) = x + yi$, then $x + y = \dots\dots\dots$

- (a) 4 (b) 3 (c) 2 (d) 1

(9) If x, y are real numbers and $x + yi = i^{43} + 3\sqrt{-4}$, then $x + y = \dots\dots\dots$

- (a) 3 (b) 5 (c) $3 + 2i$ (d) $5i$



Discover the error

12 Find the simplest form of the expression : $(2 + 3i)^2(2 - 3i)$

Ahmed's answer

$$\begin{aligned} & (2 + 3i)(2 + 3i)(2 - 3i) \\ &= (2 + 3i)(4 - 9i^2) \\ &= (2 + 3i)(4 + 9) \\ &= 13(2 + 3i) \\ &= 26 + 39i \end{aligned}$$

Karim's answer

$$\begin{aligned} & (2 + 3i)^2(2 - 3i) \\ &= (4 + 9i^2)(2 - 3i) \\ &= (4 - 9)(2 - 3i) \\ &= -5(2 - 3i) \\ &= -10 + 15i \end{aligned}$$

Which of the two answers is correct? Why?



Problems that measure high standard levels of thinking

13 Choose the correct answer from those given :

(1) If L, M are the roots of a quadratic equations : $x^2 + 1 = 0$, then $L^{2018} + M^{2018} = \dots\dots\dots$

- (a) $-2i$ (b) $2i$ (c) -2 (d) 2018

(2) $(1 + i)^{2020} = \dots\dots\dots$

- (a) $(1 - i)^{2020}$ (b) 2^{1010} (c) $2^{1010}i$ (d) i^{2020}

(3) If $\left(\frac{1-i}{1+i}\right)^{100} = x + yi$, then $(x, y) = \dots\dots\dots$

- (a) $(0, 1)$ (b) $(-1, 0)$ (c) $(0, -1)$ (d) $(1, 0)$

(4) The conjugate of the number $(2 + i)^{-1}$ is $\dots\dots\dots$

- (a) $2 + i$ (b) $2 - i$ (c) $\frac{2-i}{5}$ (d) $\frac{2+i}{5}$

(5) Which of the following considering factorization of the expression : $x^2 + 4$?

- (a) $(x-2)(x+2)$ (b) $(x+2)^2$
(c) $(x-2i)^2$ (d) $(x-2i)(x+2i)$

(6) To find the real value of x, y , it is sufficient to have $\dots\dots\dots$

- (a) $(x+2) + 4yi = 3 - 4i$ only (b) $(2x+y) + 5i = 7 + 5i$ only.
(c) (a), (b) together. (d) Nothing of the previous.

(7) The smallest positive integer (n) which makes $\left(\frac{1+i}{1-i}\right)^n = 1$ is $\dots\dots\dots$

- (a) 2 (b) 4 (c) 8 (d) 12

(8) If a, b, c, d are four positive consecutive integers : $i^a + i^b + i^c + i^d = \dots\dots\dots$

- (a) zero (b) -1 (c) 1 (d) i

(9) $i + i^2 + i^3 + i^4 + \dots + i^{100} = \dots\dots\dots$

- (a) i (b) -1 (c) zero (d) $i^{1+2+3+\dots}$

(10) $(1+i)(1+i^2)(1+i^3)(1+i^4)\dots(1+i^{100}) = \dots\dots\dots$

- (a) 2 (b) 1 (c) zero (d) Nothing of the previous

(11) If $i^m = i^n$, then which of the following is always correct?

- ① $m = n$
② $(m+n)$ is an even number
③ $(n-m)$ is multiple of 4
(a) ① only (b) ①, ③ only.
(c) ②, ③ only. (d) All the previous.

(12) If $a < b < 0 < c$ where a, b, c are real numbers and $\sqrt{b(c-a)} + \sqrt{ab} = 2 + 3i$, then $bc = \dots\dots\dots$

- (a) 3 (b) -3 (c) 2 (d) -5

(13) Which of the following is true?

- (a) $2 + 3i < 3 + 4i$ (b) $3 - 4i < 2 - 3i$
(c) $1 + i > -1 - i$ (d) Nothing of the previous.

14 If $7i = (x + 3i)(y - i) - 9$, find the values of the two real numbers x and y which satisfy the previous equation.

15 If $x = \frac{2+i}{2-i}$, $y = \frac{2+3i}{2+i}$ and $2x - y = a + bi$, prove that : $9a^2 + b^2 = 1$

2

Determining the types of roots of a quadratic equation



Test

From the school book

yourself

1 Determine the type of the two roots of each of the following equations :

(1) $x^2 - 2x + 5 = 0$

(2) $x^2 - 11x + 10 = 0$

(3) $x^2 - 10x + 25 = 0$

(4) $49x^2 - 14x + 1 = 0$

(5) $-x^2 + 5x - 30 = 0$

(6) $2x^2 - 7x = 0$

2 Determine the type of the two roots of each of the following equations :

(1) $x(x - 2) = 5$

(2) $6x^2 = 19x - 15$

(3) $(x - 11) - x(x - 6) = 0$

(4) $(x + 2)^2 + 5 = 0$

(5) $x + \frac{9}{x} = 6$

(6) $x - \frac{2}{x-1} = 4$

(7) $\frac{x}{x+1} + \frac{x}{x-1} = 3$

(8) $(x - 1)(x - 7) = 2(x - 3)(x - 4)$

3 Prove that : The two roots of the equation : $2x^2 - 3x + 2 = 0$ are complex and not real , then use the general formula to find those two roots.

4 If the two roots of each of the following quadratic equations are equal , then find the value of k :

(1) $3x^2 - 6x + k = 0$

« 3 »

(2) $18x^2 - kx + 8 = 0$

« ± 24 »

(3) $2x^2 + 5x + 4k = 0$

« $\frac{25}{32}$ »

(4) $75x^2 + 7kx + 3 = 0$

« $\pm \frac{30}{7}$ »

(5) $x^2 - 3x + 2 + \frac{1}{k} = 0$

« 4 »

(6) $x^2 + (2k + 3)x + k^2 = 0$

« $-\frac{3}{4}$ »

(7) $x^2 + 2(k - 1)x + (2k + 1) = 0$, then find the two roots.

« 0 , 1 , 1 or 4 , -3 , -3 »

(8) $x^2 - 2kx + 7k - 6x + 9 = 0$, then find the two roots.

« 0 , 3 , 3 or 1 , 4 , 4 »

5 Find the value of k in each of the following cases :

(1) If the two roots of the equation : $x^2 + 4x + k = 0$ are real and different.

« $k \in]-\infty, 4[$ »

(2) If the two roots of the equation : $kx^2 - 8x + 16 = 0$ are complex and not real.

« $k \in]1, \infty[$ »

(3) If the equation : $x^2 = k + 2$ has two real and different roots.

« $k \in]-2, \infty[$ »

6 Find the values of the real number m that make the equation :

$(m - 1)x^2 - 2mx + m = 0$ has no real roots.

« $m \in]-\infty, 0[$ »

7 Without solving any of the following equations , show which of them has two rational roots and which of them doesn't have rational roots , then check your answer by solving the equation :

(1) $2x^2 - 3x - 2 = 0$

(2) $x^2 + \sqrt{5}x - 5 = 0$

(3) $2(x + 3) + x(x - 1) = 9$

8 If a and b are rational numbers , prove that the two roots of the equation :

$ax^2 + bx + b - a = 0$ are rational.

9 If L and M are two rational numbers , then prove that the two roots of the equation :

$Lx^2 + (L - M)x - M = 0$ are rational numbers.

10 Prove that the two roots of the equation :

$x^2 + kx + k = 1$ are always rational where $k \in \mathbb{Q}$

11 If a and b are two rational numbers , prove that the two roots of the equation :

$x^2 - 2a^3x + a^6 - b^6 = 0$ are rational numbers.

12 Find the interval to which a belongs that makes the two roots of the equation :

$(a + 2)x^2 + (2a + 3)x + a - 1 = 0$ real numbers.

« $a \in [-\frac{17}{8}, \infty[$ »

13 Prove that for all the real values of a except zero the equation :

$(a^2 + 1)x^2 - 2a^3x + a^4 = 0$ has no real roots.

- 14** Prove that for all real values of a and b , the roots of the equation :

$$(X - a)(X - b) = 5 \text{ are real.}$$

- 15** Prove that for all real values of a except ($a = 2$) the equation :

$$(a - 1)X^2 - aX + 1 = 0 \text{ has two real and different roots.}$$

Problems that measure high standard levels of thinking

- 16** Choose the correct answer from those given :

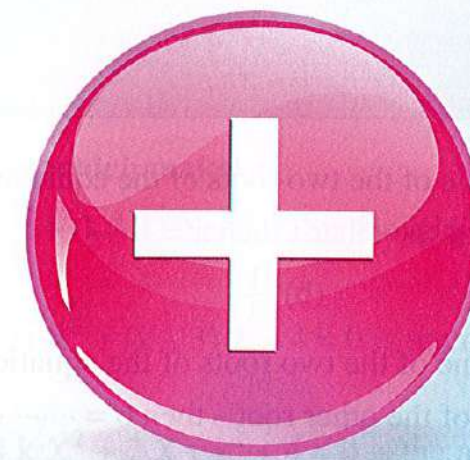
- (1) The two roots of the equation $X^2 - 2\sqrt{5}X + 1 = 0$ are
 (a) real and rational. (b) not real.
 (c) real and equal. (d) real and irrational.
- (2) If $aX^2 + bX + c = 0$, $a \in \mathbb{R}^*$, $b \in \mathbb{R}$, $c \in \mathbb{R}$ and $(b^2 - 4ac)$ is non-positive, then the two roots of the equation are
 (a) equal. (b) not real.
 (c) complex and conjugate to each other. (d) real and different.
- (3) If the roots of the equation $aX^2 + b = 0$ are real and different then
 (a) $ab > 0$ (b) $a = 0$ (c) $ab < 0$ (d) $a > 0, b > 0$
- (4) If a, b, c are real numbers, $a + b + c = 0$, $a \neq c$, then the two roots of the equation $(b + c - a)X^2 + (c + a - b)X + (a + b - c) = 0$ are
 (a) real and equal. (b) real different and rational.
 (c) real different and irrational. (d) not real.
- (5) In which of the following quadratic equations the roots are conjugate complex ?
 (a) $X^2 - 4X - 5 = 0$ (b) $\sqrt{3}X^2 + \sqrt{5}X - 1 = 0$
 (c) $X^2 - 3\sqrt{2}X + 4 = 0$ (d) $3X^2 - \sqrt{7}X + 5 = 0$
- (6) If the roots of the equation $X^2 - 2\sqrt{2}X + a = 0$ are conjugate complex, then $a \in$
 (a) $[-2, 2]$ (b) $]-\infty, 2]$
 (c) $]2, \infty[$ (d) $[2, \infty[$

- 17** If a, b and c are real numbers, then prove that the two roots of the equation :

$$X^2 + 2aX + a^2 = b^2 + c^2 \text{ are real.}$$

- 18** Prove that the two roots of the equation :

$$\frac{1}{X+a} = \frac{1}{X} + \frac{1}{a} \text{ are always not real if } a \in \mathbb{R}^*, X \notin \{0, -a\}$$




Exercise

3

Relation between the two roots of the second degree equation and the coefficients of its terms ?




Test
yourself

 From the school book

- 1** Without solving the equation, find the sum and the product of the two roots of each of the following equations :

(1) $X^2 + 3X - 10 = 0$


(2) $X^2 - 5X + 6 = 0$

(3)  $4X^2 + 4X - 35 = 0$

(4) $2X^2 - 7X - 6 = 0$

(5) $3X^2 - 4 = 0$

(6) $X^2 - 3X = 0$

(7)  $3X^2 = 23X - 30$

(8) $(4X + 1)(X + 6) = (X - 2)(3X - 4)$

(9) $\frac{X}{2} + \frac{1}{X} = \frac{3}{2}$

(10) $\frac{3X + 2}{X + 2} = \frac{X + 1}{X - 1}$

(11) $(a - 1)X^2 + X - a^2X - 1 + a = 0$

(12) $(a + b)X^2 + (a^2 - b^2)X + a^2 + 2ab + b^2 = 0$

- 2** Choose the correct answer from the given ones :

- (1) The product of the two roots of the equation : $3 + 2X - \frac{1}{4}X^2 = 0$ equals
 (a) $-\frac{2}{3}$ (b) 12 (c) -12 (d) $\frac{3}{4}$

- (2) The equation : $bX^2 + cX + a = 0$, the product of its two roots equals
 (a) $-\frac{c}{a}$ (b) $\frac{a}{b}$ (c) $-\frac{c}{b}$ (d) $\frac{a}{c}$

- (3) The sum of the two roots of the equation : $5X^2 - 3 = 0$ is
 (a) $\frac{3}{5}$ (b) $-\frac{3}{5}$ (c) zero (d) $\frac{5}{3}$

- (4) If $L, (3 - L)$ are the two roots of the equation : $X^2 - aX - 8 = 0$, then $a =$
 (a) 3 (b) -3 (c) 8 (d) -8

- (5) If one of the two roots of the equation : $x^2 - 3x + 2 = 0$ is the multiplicative inverse of the other, then $a = \dots\dots\dots$
- (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 2 (d) 3
- (6) If one of the two roots of the equation : $x^2 - (b-3)x + 5 = 0$ is the additive inverse of the other root, then $b = \dots\dots\dots$
- (a) -5 (b) -3 (c) 3 (d) 5
- (7) In the equation : $ax^2 + bx + c = 0$, if the sum of the two roots = the product of the two roots, then $b = \dots\dots\dots$
- (a) a (b) $-a$ (c) c (d) $-c$
- (8) If the product of the two roots of the equation : $(k-2)x^2 - 6x + 12 = 0$ is 3, then $k = \dots\dots\dots$
- (a) zero (b) 4 (c) 6 (d) 38
- (9) If one root of the equation : $3x^2 - (k+2)x + k^2 + 2k = 0$ is the multiplicative inverse of the other, then $k = \dots\dots\dots$
- (a) -3 or 1 (b) -3 or -1 (c) 3 or -1 (d) 3 or 1
- 3 If the product of the two roots of the equation : $3x^2 + 10x - c = 0$ is $-\frac{8}{3}$, find the value of c , then solve the equation in the set of complex numbers. « $c = 8, x = \frac{2}{3}$ or $x = -4$ »
- 4 If the sum of the two roots of the equation : $2x^2 + bx - 5 = 0$ is $-\frac{3}{2}$, find the value of b , then solve the equation in the set of complex numbers. « $b = 3, x = -\frac{5}{2}$ or $x = 1$ »
- 5 Find the other root of the equation, then find the value of a in each of the following where $a \in \mathbb{R}$:
- (1) If $x = -1$ is one of the two roots of the equation : $x^2 - 2x + a = 0$ « 3, -3 »
- (2) If $x = \frac{1}{2}$ is one of the two roots of the equation : $2x^2 - ax + 3 = 0$ « 3, 7 »
- (3) If $(1+i)$ is one of the two roots of the equation : $x^2 - 2x + a = 0$ « $1-i, 2$ »
- (4) If $(2+i)$ is one of the two roots of the equation : $x^2 + ax + 5 = 0$ « $2-i, -4$ »
- 6 Find the values of a, b in each of the following equations, if :
- (1) 2, 5 are the two roots of the equation : $x^2 + ax + b = 0$ « $a = -7, b = 10$ »
- (2) -3, 7 are the two roots of the equation : $ax^2 - bx - 21 = 0$ « $a = 1, b = 4$ »
- (3) -1, $\frac{3}{2}$ are the two roots of the equation : $ax^2 - x + b = 0$ « $a = 2, b = -3$ »
- (4) $\sqrt{3}i, -\sqrt{3}i$ are the two roots of the equation : $x^2 + ax + b = 0$ « $a = 0, b = 3$ »

7 Find the value of k in each of the following which makes :

- (1) One of the roots of the equation : $x^2 + (k-1)x - 3 = 0$ is the additive inverse of the other roots. « 1 »
- (2) One of the roots of the equation : $(k-2)x^2 + (k-3)x - 4 = 0$ is the multiplicative inverse of the other root. « -2 »
- (3) One of the roots of the equation : $4kx^2 + 7x + k^2 + 4 = 0$ is the multiplicative inverse of the other. « 2 »
- (4) One of the roots of the equation : $2x^2 + k^2 = 5x + 2$ is the multiplicative inverse of the other root. « ± 2 »
- 8 Find the value of c , if one of the two roots of the equation : $x^2 - 3x + c = 0$ is double the other root. « 2 »
- 9 Find the value of k , if one of the two roots of the equation : $x^2 + kx - 98 = 0$ is double the additive inverse of the other. « ± 7 »
- 10 Find the value of n which makes one of the two roots of the equation : $x^2 - 5x + n = 0$ exceeds the other root by 1 « 6 »
- 11 Find the value of a which makes one of the two roots of the equation : $x^2 - ax + 21 = 0$ exceeds double the other root by one. « -9.5 or 10 »
- 12 In the equation $(a-2)x^2 + (a-3)x - 4 = 0$, find the value of a if :
- (1) The sum of its roots equals 3
- (2) The product of its roots equals -4 « $\frac{9}{4}, 3$ »
- 13 In the equation $(k-4)x^2 - (3-k)x - 3 = 0$, find the value of k if :
- (1) The sum of its two roots equals 5
- (2) The product of its two roots equals -3
- (3) One of its two roots equals the additive inverse of the other root.
- (4) One of its two roots equals the multiplicative inverse of the other root. « $\frac{23}{6}, 5, 3, 1$ »
- 14 Find the value of k which makes one of the two roots of the equation : $2x^2 - (k-1)x + (k^2 + 2k - 3) = 0$ double the other root. « -3.5 or 1 »
- 15 Find the value of a which makes one of the two roots of the equation : $x^2 - ax + 2a - 4 = 0$ four times the other root. « 10 or $2\frac{1}{2}$ »

- 16** If the sum of the two roots of the equation : $(a-2)x^2 - ax + b^2 = 0$ equals 3 and the product of the roots is 5, find the value of each of a, b « 3, $\pm\sqrt{5}$ »
- 17** Find the value of c which makes one of the two roots of the equation : $x^2 - 6x + c = 0$ equals the square of the other root. « -27 or 8 »
- 18** If one of the two roots of the equation : $8x^2 - 30x + c = 0$ equals the square of the other root, find the value of c « 27 or -125 »
- 19** Find the value of a which makes one of the two roots of the equation : $4x^2 - ax - 3 = 0$ exceeds the additive inverse of the other root by 1 « 4 »
- 20** Find the value of a which makes one of the two roots of the equation : $2x^2 - ax + 3 = 0$ exceeds the multiplicative inverse of the other root by 1 « 7 »
- 21** Find the value of c , if one of the two roots of the equation : $x^2 - 10x + c = 0$ is less by 2 than the square of the other root. « -56 or 21 »
- 22** If the ratio between the two roots of the equation : $ax^2 + bx + c = 0$ as the ratio 2 : 3, prove that : $25ac = 6b^2$
- 23** If the two roots of the equation : $8x^2 - bx + 3 = 0$ are positive and the ratio between them is 2 : 3, find the value of b « 10 »
- 24** If the sum of the two roots of the equation : $(a+1)x^2 + (3a-1)x + a^2 + 1 = 0$ equals the product of its roots, find the value of a « 0 or -3 »
- 25** Find the satisfying condition such that one of the two roots of the equation $ax^2 + bx + c = 0$:
- (1) Is double the other root.
- (2) Exceeds the other root by 3 « $9ac = 2b^2, 4ac = b^2 - 9a^2$ »
- 26** Find the value of a which makes the sum of the two roots of the equation : $x^2 - (a+4)x + 3a^2 = 0$ equals the product of the two roots of the equation : $2x^2 - 7ax + a^2 = 0$ « 4 or -2 »



Discover the error

- 27** If the product of the two roots of the equation : $x^2 + 4x + k = 2$ is 12, find the value of k

Mona's answer

\therefore Product of the two roots = 12
 $\therefore \frac{k}{1} = 12$
 $\therefore k = 12$

Noura's answer

$\therefore x^2 + 4x + k = 2$
 $\therefore x^2 + 4x + k - 2 = 0$
 \therefore Product of the two roots = 12
 $\therefore \frac{k-2}{1} = 12 \therefore k - 2 = 12 \therefore k = 14$

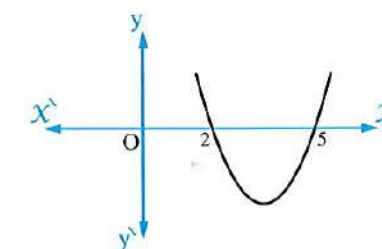
Which answer is correct? Why?



Problems that measure high standard levels of thinking

- 28** Choose the correct answer from those given :

- (1) If (2 i) is one root of the equation : $x^2 + ax + b = 0$ where coefficients of its terms are real numbers, then all of the following are true except
- (a) the other root is (-2 i) (b) sum of the two roots = zero
 (c) product of the two roots = -4 (d) discriminant of the equation < 0
- (2) To evaluate the real values of b, c in the equation : $x^2 + bx + c = 0$, it is sufficient to have
- (a) real roots sum = 6 only. (b) one of the roots = (3 + i) only.
 (c) (a), (b) together. (d) nothing of the previous.
- (3) If the opposite figure Represents the curve of the function $f : f(x) = ax^2 + bx + c$, then $\frac{b+c}{a} = \dots\dots\dots$
- (a) 3 (b) 5
 (c) 7 (d) 10
- (4) The product of the roots of the equations : $ax^2 + bx + c = 0$, $bx^2 + cx + a = 0$, $cx^2 + ax + b = 0$ equals
- (a) abc (b) -1 (c) 1 (d) zero
- (5) If x_1, x_2 are the roots of the equation : $ax^2 + bx + c = 0$ and $x_1 < 0 < x_2$, $|x_1| > |x_2|$, which of the following statements could be true?
- (a) $a < 0$ (b) $bc > 0$ (c) $bc < 0$ (d) $x_1 + x_2 > 0$



- 29** Find the value of a which makes the two roots of the equation :

$3x^2 - (2a-1)x + (a-4) = 0$ are different in sign.

« $a \in]-\infty, 4[$ »

4

Forming the quadratic equation
whose two roots are knownTest
yourself

From the school book

1 Choose the correct answer from those given :

(1) The quadratic equations whose roots sum -1 and their product -3 is

- (a) $x^2 - x - 3 = 0$ (b) $x^2 + x + 3 = 0$
 (c) $x^2 - x + 3 = 0$ (d) $x^2 + x - 3 = 0$

(2) The quadratic equation whose roots are $3, -5$ is

- (a) $x^2 + 2x - 15 = 0$ (b) $x^2 - 2x - 15 = 0$
 (c) $x^2 - 2x + 15 = 0$ (d) $x^2 + 2x + 15 = 0$

(3) If L, L^2 are the roots of the equation : $2x^2 + bx + 54 = 0$, then $b =$

- (a) -12 (b) -24 (c) 27 (d) 36

(4) If $m, \frac{2}{m}$ are the roots of the equation : $ax^2 + bx + 12 = 0$, then $a =$

- (a) 3 (b) 5 (c) 6 (d) 9

2 Form the quadratic equation whose two roots are :

- | | | |
|---------------------------------------|--|---|
| (1) $-2, 4$ | (2) $7, 7$ | (3) $-7, 0$ |
| (4) $\frac{2}{3}, \frac{3}{2}$ | (5) $\frac{3}{5}, -2\frac{1}{5}$ | (6) $5\sqrt{3}, -2\sqrt{3}$ |
| (7) $7 + 2\sqrt{5}, 7 - 2\sqrt{5}$ | (8) $-5i, 5i$ | (9) $1 - 3i, 1 + 3i$ |
| (10) $3 - 2\sqrt{2}i, 3 + 2\sqrt{2}i$ | (11) $\frac{3}{i}, \frac{3+3i}{1-i}$ | (12) $\frac{-2+2i}{1+i}, \frac{-2-4i}{2-i}$ |
| (13) $a-b, a+b$ | (14) $\frac{a^2-b^2}{a-b}, \frac{a^3-b^3}{a^2+ab+b^2}$ | |

3 If L and M are the two roots of the equation : $x^2 - 7x + 5 = 0$,
 then find the numerical value of each of the following expressions :

- | | | |
|---------------------|--|---|
| (1) $L^2 M + M^2 L$ | (2) $\frac{1}{M} + \frac{1}{L}$ | |
| (3) $(L-2)(M-2)$ | (4) $(L + \frac{1}{M})(M + \frac{1}{L})$ | $\ll 35, \frac{7}{5}, -5, 7\frac{1}{5} \gg$ |

4 If L and M are the two roots of the equation : $x^2 - 4x + 2 = 0$, where $L > M$,
 find the numerical value of each of the following expressions :

- | | | |
|--------------------|----------------------|------------------------------------|
| (1) $L^2 + M^2$ | (2) $L - M$ | (3) $L^3 + M^3$ |
| (4) $L^2 - 4L + 7$ | (5) $2M^2 - 8M + 15$ | $\ll 12, 2\sqrt{2}, 40, 5, 11 \gg$ |

5 If L and M are the two roots of the equation : $x^2 - 3x - 5 = 0$, then find the equation
 whose roots are : $L - 4$ and $M - 4$ $\ll x^2 + 5x - 1 = 0 \gg$

6 If L and M are the two roots of the equation : $x^2 - 5x + 3 = 0$, then find the equation
 whose roots are : $2L$ and $2M$ $\ll x^2 - 10x + 12 = 0 \gg$

7 If L and M are the two roots of the equation : $2x^2 - 5x - 7 = 0$, then find the equation
 whose roots are : $1 - L$ and $1 - M$ $\ll 2x^2 + x - 10 = 0 \gg$

8 If L and M are the two roots of the equation : $x^2 - 3x + 1 = 0$, then form the equation
 whose roots are : $L + M$ and LM $\ll x^2 - 4x + 3 = 0 \gg$

9 If L and M are the two roots of the equation : $x^2 - 3x - 4 = 0$, then find the equation
 whose roots are : $\frac{1}{L}$ and $\frac{1}{M}$ $\ll 4x^2 + 3x - 1 = 0 \gg$

10 If L and M are the two roots of the equation : $2x^2 - 3x - 6 = 0$, then form the
 equation whose roots are : $\frac{L}{4}$ and $\frac{M}{4}$ $\ll 16x^2 - 6x - 3 = 0 \gg$

11 If L and M are the roots of the equation : $2x^2 - 5x + 1 = 0$, then find the equation
 whose roots are : $2L^2$ and $2M^2$ $\ll 2x^2 - 21x + 2 = 0 \gg$

12 Find the quadratic equation in which each of the two roots exceeds one of the two
 roots of the equation : $x^2 - 7x - 9 = 0$ $\ll x^2 - 9x - 1 = 0 \gg$

13 Form the quadratic equation in which each of its two roots equals half of its corresponding
 root of the equation : $4x^2 - 12x + 7 = 0$ $\ll 16x^2 - 24x + 7 = 0 \gg$

- 14 Find the quadratic equation in which each of its two roots equals the square of the corresponding root of the equation : $x^2 + 3x - 5 = 0$ « $x^2 - 19x + 25 = 0$ »
- 15 If L and M are the two roots of the equation : $2x^2 - 3x - 1 = 0$, then form the quadratic equations whose two roots are : $\frac{L}{M}, \frac{M}{L}$ « $2x^2 + 13x + 2 = 0$ »
- 16 If L and M are the two roots of the equation : $x^2 + 5x + 6 = 0$, find the equation whose roots are : L - M and M - L « $x^2 - 1 = 0$ »
- 17 If L and M are the two roots of the equation : $x^2 - 2x - 4 = 0$, find the equation whose roots are : $\frac{1}{L^2}$ and $\frac{1}{M^2}$ « $16x^2 - 12x + 1 = 0$ »
- 18 If L and M are the two roots of the equation : $3x^2 - 5x + 2 = 0$, form the equation whose roots are : $\frac{L^2}{M}$ and $\frac{M^2}{L}$ « $18x^2 - 35x + 12 = 0$ »
- 19 If L and M are the two roots of the equation : $10x^2 + 12x - 1 = 0$, form the equation whose roots are : $2L + \frac{1}{M}$, $2M + \frac{1}{L}$ « $5x^2 - 48x - 32 = 0$ »
- 20 If L and M are the two roots of the equation : $x^2 - 3x - 5 = 0$, find the equation whose roots are : $L^2 M$ and $M^2 L$ « $x^2 + 15x - 125 = 0$ »
- 21 If L and M are the two roots of the equation : $x^2 - 3x + 5 = 0$, find the equation whose roots are : $6, L^2 + M^2$ « $x^2 - 5x - 6 = 0$ »
- 22 If L and M are the two roots of the equation : $x^2 - 3x - 1 = 0$, where $L > M$, form the equation whose roots are : $3L - 2M, 2L - 3M$ « $x^2 - 5\sqrt{13}x + 79 = 0$ »
- 23 If L + 2 and M + 2 are the two roots of the equation : $x^2 - 11x + 3 = 0$, find the equation whose roots are : L , M « $x^2 - 7x - 15 = 0$ »
- 24 If L + 3 and M + 3 are the two roots of the equation : $x^2 - 5x + 11 = 0$, form the equation whose roots are : $L^2 M$ and $M^2 L$ « $x^2 + 5x + 125 = 0$ »
- 25 If $\frac{1}{L}, \frac{1}{M}$ are the two roots of the equation : $x^2 - 3x + 1 = 0$, form the equation whose roots are : $LM - 7, L + M + 3$ « $x^2 - 36 = 0$ »
- 26 If L and M are the two roots of the equation : $x^2 - 2x - 5 = 0$, form the equation whose roots are : $L^2 + M, M^2 + L$ « $x^2 - 16x + 58 = 0$ »

- 27 If $\frac{3}{L}$ and $\frac{3}{M}$ are the two roots of the equation : $x^2 - 12x + 9 = 0$, form the equation whose roots are : $\frac{1}{L^3}, \frac{1}{M^3}$ « $x^2 - 52x + 1 = 0$ »
- 28 If the difference between the two roots of the equation : $6x^2 - 7x + 1 = c$ is $\frac{11}{6}$, find the value of c « 4 »
- 29 If the difference between the two roots of the equation : $3x^2 - 2x + c = 0$ equals the difference between the two roots of the equation : $2x^2 - cx + 3 = 0$, prove that : $9c^2 + 48c - 232 = 0$
- 30 If the difference between the two roots of the equation : $x^2 + kx + 2k = 0$ equals twice the product of the two roots of the equation : $x^2 + 3x + k = 0$, then find the value of k « 0 or $-\frac{8}{3}$ »
- 31 If L and M are the two roots of the equation : $4x^2 - 6x + a = 0$ and $L^2 + M^2 = 7LM$, find the value of a « 1 »
- 32 If L and M are the two roots of the equation : $x^2 - 8x + c = 0$ and $L^2 + M^2 = 40$, find the numerical value of c , then form the equation whose roots are : $L^2 M + M^2 L, LM$ « $c = 12, x^2 - 108x + 1152 = 0$ »
- 33 If L and M are the two roots of the equation : $x^2 - 4x - 5 = 0$, where $L > M$, then form the equation whose roots are : $L - 7, 2M^2 + 1$ « $x^2 - x - 6 = 0$ »



Discover the error

- 34 If L + 1 and M + 1 are the roots of the equation : $x^2 + 5x + 3 = 0$, then find the quadratic equation whose roots are : L and M

Yousef's answer

$$\begin{aligned} \because (L+1) + (M+1) &= -5 \\ \therefore L+M+2 &= -5 \\ \therefore L+M &= -7 \\ \because (L+1)(M+1) &= 3 \\ \therefore LM + (L+M) + 1 &= 3 \\ \therefore LM - 7 + 1 &= 3 \\ \therefore LM &= 9 \\ \therefore \text{The equation is : } x^2 + 7x + 9 &= 0 \end{aligned}$$

Amira's answer

$$\begin{aligned} \because L+M &= -5 \\ , LM &= 3 \\ \therefore (L+1) + (M+1) &= L+M+2 = -5+2 = -3 \\ , \because (L+1)(M+1) &= LM + (L+M) + 1 \\ &= 3 - 5 + 1 = -1 \\ \therefore \text{The equation is : } x^2 + 3x + 1 &= 0 \end{aligned}$$

Which of the two answers is correct ? Why ?

Problems that measure high standard levels of thinking

35 Choose the correct answer from those given :

(1) The quadratic equation whose roots are the dimensions of a rectangle of area 15 cm^2 and its perimeter 26 cm . is

- (a) $x^2 - 26x + 15 = 0$ (b) $x^2 + 26x - 15 = 0$
(c) $x^2 - 13x - 15 = 0$ (d) $x^2 - 13x + 15 = 0$

(2) If $a^2 + 3a + 1 = 0$, $b^2 + 3b + 1 = 0$ where a, b are real different numbers, then $\frac{a}{b} + \frac{b}{a} = \dots\dots\dots$

- (a) 2 (b) 7 (c) -5 (d) 11

(3) If L, M are the roots of the quadratic equation : $(x - a)(x - b) = k$, then the quadratic equation whose roots are a and b is

- (a) $(x - L)(x - m) = 0$ (b) $(x - L)(x - m) + k = 0$
(c) $(x - L)(x - m) = k$ (d) $x^2 - (L + m)x + k = 0$

(4) To form the quadratic equation whose roots $4L, 4M$ where L, M are real numbers it is sufficient to have

- (a) $L + M = 5$ only. (b) $(L + M + 4)^2 + (LM - 3)^2 = \text{zero only.}$
(c) (a), (b) together. (d) nothing of the previous.

(5) Omar and Khaled are trying to solve a quadratic equation Omar miswrite the absolute term of the equation and he got the roots of the equation $3, 4$, while Khaled miswrite the coefficient of x in the equation so he got the roots of the equation $2, 3$ then the right roots of the equation are

- (a) $2, 4$ (b) $-2, -4$ (c) $1, 6$ (d) $-1, -6$

(6) If the roots of the quadratic equation : $x^2 + bx + c = 0$ are two consecutive odd numbers, then $b^2 - 4c = \dots\dots\dots$

- (a) -1 (b) 2 (c) 3 (d) 4

(7) If the roots of the quadratic equation : $x^2 - bx + c = 0$ are two different integers and b, c are prime numbers which of the following statements could be right ?

- ① The difference between the equation roots is odd.
② $b^2 - c$ is a prime number ③ $b + c$ is a prime number

- (a) ① only (b) ①, ③ only. (c) ②, ③ only. (d) All the previous.

(8) If the curve of the function f where $f(x) = ax^2 + bx + c$ intersects x -axis at $x = L$, $x = M$ where $|L - M| > 1$, then

- (a) $f(L + 1) > f(L) > f(L - 1)$ (b) $f(L - 1) > f(L) > f(L + 1)$
(c) $f(L) > f(L + 1) > f(L - 1)$ (d) $f(L + 1) \times f(L - 1) < 0$

(9) If L, M are the roots of the equation : $x^2 - (\tan \theta)x - 1 = 0$ and $L^2 + M^2 = 3$ where $0^\circ < \theta < 90^\circ$, then $\theta = \dots\dots\dots$

- (a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

36 If L and M are the two roots of the equation : $ax^2 + 2bx + c = 0$, $a \neq 0$, $L > M$ and $L - M = 2$, prove that :

- (1) $b^2 = a(a + c)$ (2) $L = 1 - \frac{b}{a}$

37 If the difference between the two roots of the equation : $ax^2 + bx + c = 0$, where $a \neq 0$ equals twice the sum of their multiplicative inverses

, prove that : $c^2(b^2 - 4ac) = 4a^2b^2$

Sign of a function



Test

yourself

From the school book

1 Investigate the signs of the functions which are defined by the following rules :

- | | | |
|------------------|---------------------|-------------------------------|
| (1) $f(x) = -1$ | (2) $f(x) = 2$ | (3) $f(x) = 2x$ |
| (4) $f(x) = -3x$ | (5) $f(x) = 5x - 7$ | (6) $f(x) = 3 - \frac{1}{2}x$ |

2 Determine the sign of each of the functions which are defined by the following rules , then represent your answer on the number line :

- | | |
|----------------------------|----------------------------|
| (1) $f(x) = (x-2)(x+3)$ | (2) $f(x) = (2x-3)^2$ |
| (3) $f(x) = 2x^2 + 5x - 7$ | (4) $f(x) = x^2 - 4x + 3$ |
| (5) $f(x) = x^2 - 8x + 16$ | (6) $f(x) = 2x^2 - 3x + 5$ |
| (7) $f(x) = 4x - 7 - x^2$ | (8) $f(x) = 9 - 4x^2$ |
| (9) $f(x) = 2x^2$ | |

3 Draw the curve of the function $f : f(x) = 2x^2 - 8$ in $[-2, 2]$ From the graph , determine the sign of f in \mathbb{R} 4 Draw the curve of the function $f : f(x) = 2x^2 - 3x + 4$ in $[-1, 2\frac{1}{2}]$ From the graph , determine the sign of f in \mathbb{R} 5 Draw the curve of the function $f : f(x) = -x^2 + 8x - 15$ in $[1, 7]$ From the graph , determine the sign of f in \mathbb{R} and the solution of the equation $f(x) = 0$ « {3, 5} »6 Draw the curve of the function $f : f(x) = x^2 - 9$ in the interval $[-3, 4]$ From the graph , determine the sign of f in that interval.7 Draw the curve of the function $f : f(x) = -x^2 + 2x + 4$ in $[-3, 5]$ From the graph , determine the sign of f in that interval.

8 Choose the correct answer from those given :

- (1) The function $f : f(x) = -4$ is negative in the interval
 (a) $]-\infty, 4[$ (b) $]-4, 4[$ (c) $]-\infty, \infty[$ (d) $]-2, 2[$
- (2) The function $f : f(x) = 5x - 3$ is positive at
 (a) $x > \frac{3}{5}$ (b) $x < \frac{3}{5}$ (c) $x > \frac{5}{3}$ (d) $x < \frac{-5}{3}$
- (3) If $f(x) = x + 2$ where $x \in]-4, 3[$, then f is positive at $x \in$
 (a) $]-\infty, -2[$ (b) $]-2, \infty[$ (c) $]-4, -2[$ (d) $]-2, 3[$
- (4) The function $f : f(x) = c$ has a sign always.
 (a) positive (b) negative (c) x (d) c
- (5) If $f(x) = 3x$, then the sign of the function f is negative in the interval
 (a) $]-\infty, 3[$ (b) $]3, \infty[$ (c) $]-\infty, 0[$ (d) $]-3, \infty[$
- (6) The function $f : f(x) = x^2 - 9$ is negative $\forall x \in$
 (a) $\mathbb{R} - [-3, 3]$ (b) $]-3, 3[$ (c) $]-\infty, -9[$ (d) $]-\infty, -3[$
- (7) The function $f : f(x) = x^2 + 1$ are positive for $x \in$
 (a) $]0, \infty[$ (b) $]1, \infty[$ (c) $]-\infty, 1[$ (d) \mathbb{R}
- (8) The function $f : f(x) = x^2 - 6x + 9$ is positive in the interval
 (a) $]0, \infty[$ (b) $]-\infty, 3[$ (c) $\mathbb{R} - \{3\}$ (d) $\mathbb{R} - \{0\}$
- (9) The function $f : f(x) = ax^2 + bx + c$ has one sign in \mathbb{R} at
 (a) $b^2 - 4ac > 0$ (b) $b^2 - 4ac < 0$
 (c) $b^2 - 4ac = 0$ (d) $b^2 - 4ac \geq 0$
- (10) If the function $f : f(x) = ax^2 + bx + c$ and $a < 0$ and the two roots of $f(x) = 0$ are $2, -5$, then the function f is positive in
 (a) $\{-5, 2\}$ (b) $\mathbb{R} -]-5, 2[$ (c) $]-5, 2[$ (d) $]-\infty, -5[$
- (11) To investigate the sign of the function f its sufficient if we know
 (a) the curve of the function f parallel to x -axis only.
 (b) the curve of the function f lies completely below x -axis only.
 (c) (a) and (b) together. (d) nothing of the previous.
- (12) If $f(x) = ax + b$ and $x = L$ is the root of the function $f(x) = 0$, then $f(L+1) \times f(L-1) \in$
 (a) \mathbb{R}^+ (b) \mathbb{R}^- (c) $[-1, 1]$ (d) $[-5, 5]$
- (13) Which of the following functions is positive for all values of $x \in \mathbb{R}$
 (a) $f : f(x) = x^2 + 4$ (b) $f : f(x) = 3$
 (c) $f : f(x) = (x-1)^2 + 9$ (d) All the previous.

(14) The function $f : f(x) = 12 + 4x - x^2$ is not negative in the interval

- (a) $]-2, 6[$ (b) $[-2, 6]$ (c) $\mathbb{R} -]-2, 6[$ (d) $]-\infty, \infty[$

(15) The function $f : f(x) = -(x-1)(x+2)$ is positive in the interval

- (a) $]1, 2[$ (b) $[-1, 2]$ (c) $]-2, 1[$ (d) $]-\infty, \infty[$

(16) The opposite figure represents a first degree function in x

First : The function is positive in the interval

- (a) $[2, \infty[$ (b) $]1, \infty[$
(c) $]-\infty, 2[$ (d) $]2, \infty[$

Second : The function is negative in the interval

- (a) $]-\infty, 2]$ (b) $]-2, 2]$
(c) $]-\infty, 2[$ (d) $]2, \infty[$

(17) The opposite figure represents a second degree function f in x

First : $f(x) = 0$ at $x \in$

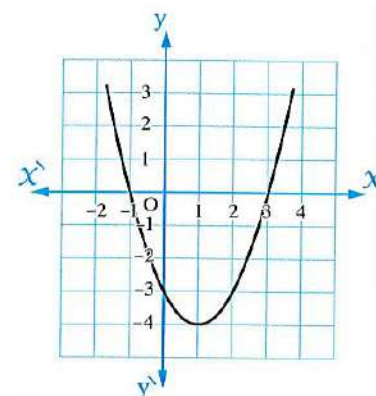
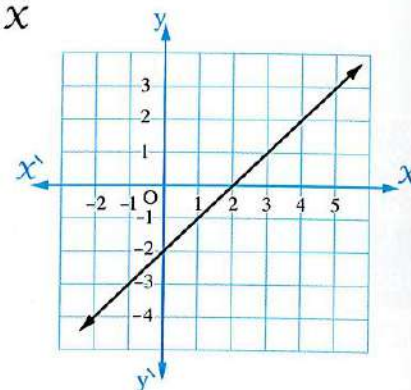
- (a) \mathbb{R} (b) \mathbb{N}
(c) $[-1, 3]$ (d) $\{3, -1\}$

Second : $f(x) > 0$ at $x \in$

- (a) $]-1, 3[$ (b) $[-1, 3]$
(c) $\mathbb{R} - [-1, 3]$ (d) \mathbb{R}

Third : $f(x) < 0$ at $x \in$

- (a) $]-1, 3[$ (b) $[-1, 3]$ (c) $\mathbb{R} - [-1, 3]$ (d) \mathbb{R}



9 Investigate the sign of each of the following functions :

(1) $f : [-1, 6] \rightarrow \mathbb{R}$ where $f(x) = 3 - x$

(2) $f : [-2, 8] \rightarrow \mathbb{R}$ where $f(x) = x^2 - 5x - 6$

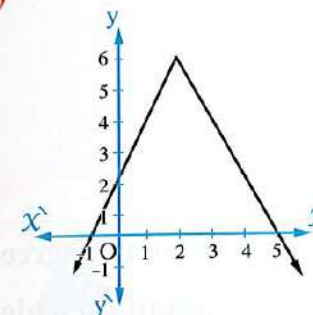
10 Investigate the sign of each of the functions which are defined by the following two rules :

$$(1) f(x) = \begin{cases} x+1 & x \geq -1 \\ -x-1 & x < -1 \end{cases}$$

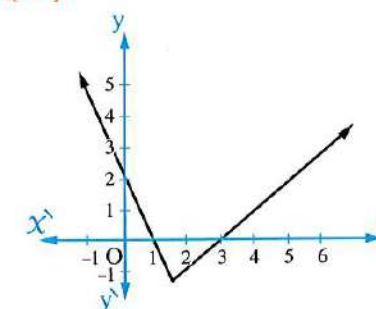
$$(2) f(x) = \begin{cases} 3x+6 & \text{at } x \geq -2 \\ -3x-6 & \text{at } x < -2 \end{cases}$$

11 Determine the sign of each of the functions represented by the following figures :

(1)



(2)



12 Determine the sign of each of the two functions : $f : f(x) = x - 3$, $g : g(x) = x^2 - 5x - 6$ and when the two functions are positive together.

13 If $f_1(x) = x - 3$, $f_2(x) = 5 + 4x - x^2$, determine the sign of each of f_1 , f_2 on the number line and determine the intervals at which the two functions are negative together.

14 If $f(x) = x^2 - 5x + 6$ and $g(x) = 2x^2 - 5x - 18$, state the two functions f , g when they are positive together or negative together.

15 Prove that for all the values of $k \in \mathbb{R}$ the two roots of the equation : $2x^2 - kx + k - 3 = 0$ are real and different.



Discover the error

16 If $f(x) = x + 1$, $g(x) = 1 - x^2$

, determine the interval at which the two functions are positive together.

Yousef's answer

$x = -1$ makes $f(x) = 0$
 $f(x)$ is positive in the interval $]-1, \infty[$
 $x = \pm 1$, makes $g(x) = 0$, $g(x)$ is positive in the interval $]-1, 1[$, thus the two functions are positive together in the interval $]-1, \infty[\cup]-1, 1[=]-1, \infty[$

Amira's answer

$x = -1$ makes $f(x) = 0$
 $f(x)$ is positive in the interval $]-1, \infty[$
 $x = \pm 1$, it makes $g(x) = 0$
 $g(x)$ is positive in the interval $]-1, 1[$ thus the two functions are positive together in the interval $]-1, \infty[\cap]-1, 1[=]-1, 1[$

Which of the two answers is correct ? Represent each of the two functions graphically and check the correct answer.

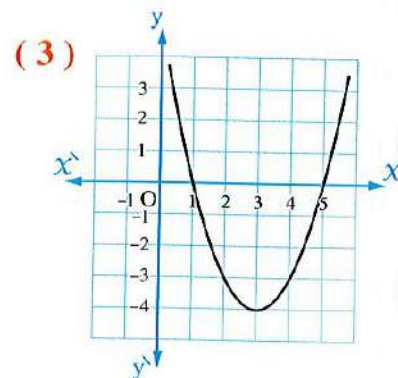
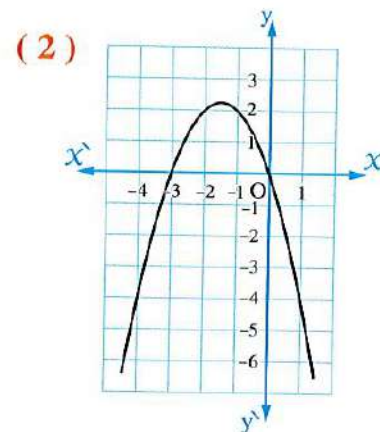
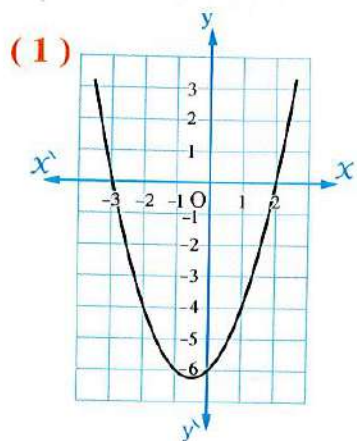
Problems that measure high standard levels of thinking

17 Study the sign of each of the following two functions :

(1) $f : f(x) = -2x^2 - 2\sqrt{2}x - 1$

(2) $f : f(x) = x + (x+1)(2x+3) - 4(x+1) + 1$

18 Each of the following figures shows the graphical representation of a second degree function in one variable. Study the sign of each function in \mathbb{R} , then find the rule of each function :



Exercise

6

Quadratic inequalities in one variable

From the school book

Test
yourself

1 Find in \mathbb{R} the solution set of each of the following inequalities :

(1) $x^2 + 2x - 8 > 0$

(2) $x^2 + 3x - 4 \geq 0$

(3) $x^2 - 5x - 6 < 0$

(4) $x^2 - x - 2 \leq 0$

(5) $4 - 3x - x^2 \geq 0$

(6) $5x - x^2 - 6 < 0$

(7) $x^2 - 1 \leq 0$

(8) $4 - x^2 < 0$

(9) $7 + x^2 - 4x < 0$

(10) $2x + x^2 + 5 > 0$

(11) $x^2 - 4x + 4 \geq 0$

(12) $6x - x^2 - 9 < 0$

(13) $x^2 - 8x + 16 < 0$

(14) $-x^2 - 10x - 25 \geq 0$

(15) $2x - x^2 < 0$

2 Find in \mathbb{R} the solution set of each of the following inequalities :

(1) $x^2 \leq 9$

(2) $x^2 > 16$

(3) $x^2 + 5x < -4$

(4) $5x^2 + 12x \geq 44$

(5) $3x^2 \leq 11x + 4$

(6) $x^2 \geq 6x - 9$

(7) $3 - 2x \geq x^2$

(8) $7x + 15 \leq 2x^2$

(9) $x^2 + 5 \leq 1$

(10) $-x^2 - 7 < 2$

(11) $(x-2)^2 \geq 9$

(12) $(x-2)^2 \leq -5$

(13) $x(x+2) - 3 \leq 0$

(14) $(x+2)^2 + (x+1)(x-4) < 0$

(15) $(x+3)^2 < 10 - 3(x+3)$

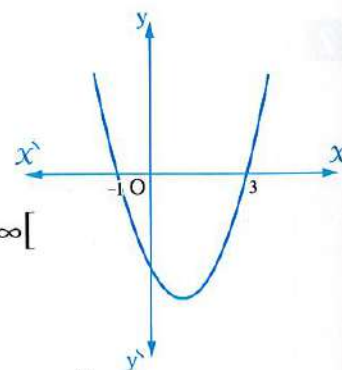
(16) $5 - 2x \leq x^2$

3 Determine the sign of the function $f : f(x) = x^2 - 5x + 6$ and from that find in \mathbb{R} the solution set of the inequality : $f(x) < 0$

4 Determine the sign of the function $f : f(x) = 2x^2 + 7x - 15$ and from that find in \mathbb{R} the solution set of the inequality : $2x^2 + 7x \leq 15$

- 5 Determine the sign of the function $f : f(x) = x^2 + 4$, then find in \mathbb{R} the solution set of the inequality : $f(x) \leq \text{zero}$
- 6 Draw the graph of the function $f : f(x) = -x^2 + 2x + 3$ in the interval $[-2, 4]$, from the graph find in \mathbb{R} :
- (1) The solution set of the equality $f(x) = 0$ (2) The solution set of the inequality $f(x) \leq 0$
- (3) The solution set of the inequality $f(x) > 0$
- 7 Choose the correct answer :

- (1) The solution set of the inequality : $(x-2)(x-5) < 0$ in \mathbb{R} is
 (a) $\{2, 5\}$ (b) $]2, 5[$ (c) $[2, 5]$ (d) $\mathbb{R} - [2, 5]$
- (2) The solution set of the inequality : $x(x-1) > 0$ in \mathbb{R} is
 (a) $\{0, 1\}$ (b) $]0, 1[$ (c) $[0, 1]$ (d) $\mathbb{R} - [0, 1]$
- (3) The solution set of the inequality : $-x(x+2) \geq 0$ in \mathbb{R} is
 (a) $\{0, -2\}$ (b) $[-2, 0]$ (c) $] -2, 0[$ (d) $[-2, 2]$
- (4) The solution set of the inequality : $x^2 + 9 > 0$ in \mathbb{R} is
 (a) \emptyset (b) \mathbb{R} (c) $] -3, 3[$ (d) $\mathbb{R} - [-3, 3]$
- (5) The solution set of the inequality : $x^2 + 1 \leq 0$ in \mathbb{R} is
 (a) \emptyset (b) \mathbb{R} (c) $[-1, 1]$ (d) $\mathbb{R} -] -1, 1[$
- (6) If the opposite figure represents the function curve $f : f(x) = x^2 - 2x - 3$, then the solution set of the inequality $x^2 - 2x - 3 \geq 0$ in \mathbb{R} is
 (a) $] -1, 3[$ (b) $] -\infty, 2[$
 (c) $] 3, \infty[$ (d) $] -\infty, -1] \cup [3, \infty[$



Discover the error

- 8 Find in \mathbb{R} the solution set of the inequality : $(x+1)^2 < 4(2x-1)^2$

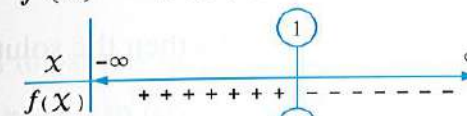
Yousef's answer

$\therefore (x+1)^2 < 4(2x-1)^2$
 $\therefore x+1 < 2(2x-1)$ by taking the square root to both sides
 $\therefore -4x + x + 2 + 1 < 0$
 $\therefore -3x + 3 < 0$
 • The equation related to the inequality is : $-3x + 3 = 0$

Nour's answer

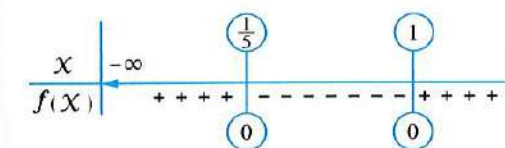
$\therefore (x+1)^2 < 4(2x-1)^2$
 $\therefore x^2 + 2x + 1 < 16x^2 - 16x + 4$
 $\therefore 15x^2 - 18x + 3 > 0$
 \therefore The equation related to the inequality is $3(5x-1)(x-1) = 0$
 \therefore The solution set = $\{1, \frac{1}{5}\}$

- The S.S. is $\{1\}$
- By investigating the sign of f where $f(x) = -3x + 3$



\therefore The solution set = $]1, \infty[$

- By investigating the sign of f where $f(x) = 15x^2 - 18x + 3$



\therefore The solution set = $\mathbb{R} - [\frac{1}{5}, 1]$

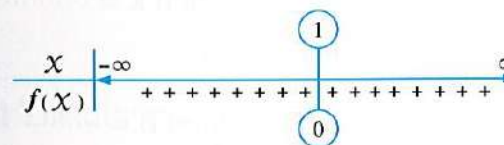
Which of the two answers is correct ?

- 9 Find in \mathbb{R} the solution set of the inequality : $x^2 - 2x + 1 \geq 0$

Basem's answer

\therefore The related equation to the inequality is $x^2 - 2x + 1 = 0 \therefore (x-1)^2 = 0$
 \therefore The S.S. = $\{1\}$

- Investigating the sign of the function f where $f(x) = x^2 - 2x + 1$

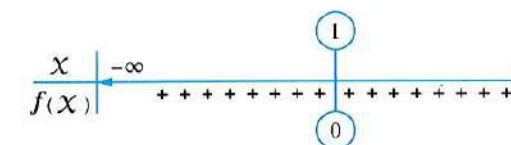


\therefore The solution set = $\mathbb{R} - \{1\}$

Eslam's answer

\therefore The related equation to the inequality is $x^2 - 2x + 1 = 0 \therefore (x-1)^2 = 0$
 \therefore The S.S. = $\{1\}$

- Investigating the sign of the function f : $f(x) = x^2 - 2x + 1$



\therefore The solution set = \mathbb{R}

Which of the two answers is correct ? Why ?

Problems that measure high standard levels of thinking

- 10 Choose the correct answer from those given :

- (1) If $f(x) = x^2 - 7x + 12$, $x \in \mathbb{R}$, then all the following are true except
 (a) solution set of the equation $f(x) = 0$ is $\{3, 4\}$
 (b) solution set of the inequality $f(x) > 0$ is $\mathbb{R} - [3, 4]$
 (c) solution set of the inequality $f(x) < 0$ is $]3, 4[$
 (d) $f(x)$ is positive in the interval $\mathbb{R} -]3, 4[$
- (2) The sum of integers belong to the solution set of the inequality $(x-2)(3x-1) \leq 0$
 (a) -1 (b) 1 (c) 2 (d) 3
- (3) The solution set of the inequality $(x+3)^2 < 4(x+1)^2$ in \mathbb{R} is
 (a) $] \frac{-5}{3}, 1[$ (b) $\mathbb{R} -] \frac{-5}{3}, 1[$ (c) $[\frac{-5}{2}, 1[$ (d) $\mathbb{R} - [\frac{-5}{3}, 1]$

(4) If L, M are the roots of the equation : $aX^2 + bX + c = 0$ where $a > 0, L < M$, then the solution set of the inequality $aX^2 + bX + c < 0$ in \mathbb{R} is

- (a) $]-\infty, L[$ (b) $]L, M[$ (c) $]M, \infty[$ (d) $\mathbb{R} - [L, M]$

(5) If the discriminant of the equation : $aX^2 + bX + c = 0$ is negative, then the solution set of the inequality $aX^2 + bX + c < 0$ where $a < 0$ in \mathbb{R} is

- (a) \mathbb{R} (b) \emptyset (c) \mathbb{R}^+ (d) \mathbb{R}^-

(6) If L, M are the two roots of the equation : $2X^2 + (k-2)X - 5 = 0$ and $-1 < L < M$, then

- (a) $-1 < k < 0$ (b) $k > 6$ (c) $k < -1$ (d) $-1 < k < 6$

(7) If each one of the two roots of a quadratic equation : $X^2 - 2kX + k^2 + k - 5 = 0$ is less than 5, then $k \in$

- (a) $[4, 5]$ (b) $[4, \infty[$ (c) $]-\infty, 4[$ (d) $\mathbb{R} - [4, 5]$

(8) If the two roots of the quadratic equation : $X^2 - kX + 1 = 0$ are not real, then

- (a) $k \in \mathbb{Z}^-$ (b) $-2 < k < 2$ (c) $k > 2$ (d) $k < -2$

(9) If the solution set of the inequality : $X^2 - 4 \leq X + k$ is $[-2, 3]$, then $k =$

- (a) -6 (b) 1 (c) 2 (d) 10

(10) If the solution set of the inequality : $X^2 - 10 < bX$ is $]-2, 5[$, then $b =$

- (a) -10 (b) -2 (c) 3 (d) 5

(11) If one of the roots of the equation : $X^2 - bX + 3 = 0$ belongs to the interval $]1, 2[$, then $b \in$

- (a) $]1, 2[$ (b) $]-\infty, 3[$ (c) $]3\frac{1}{2}, 4[$ (d) $\mathbb{R} -]3\frac{1}{2}, 4[$

(12) If S_1 is the solution set of the inequality : $X^2 - X - 2 \leq 0$ and S_2 is the solution set of the inequality : $X^2 + X - 2 \leq 0$, then $S_1 \cap S_2 =$

- (a) \emptyset (b) $[-2, 2]$ (c) $[-1, 1]$ (d) $\mathbb{R} -]-1, 1[$

(13) If L, M are the roots of the equation : $aX^2 + aX + a + 2 = 0$ and $2 \in]L, M[$, then $a \in$

- (a) $[1, 2]$ (b) \mathbb{R}^+ (c) $]-\frac{2}{7}, 0[$ (d) $]\frac{2}{L}, \frac{2}{M}[$

(14) If the two roots of the quadratic equation : $4X^2 - 2X + m = 0$ belong to the interval $]-1, 1[$, then

- (a) $0 \leq m < 2$ (b) $-6 < m < \frac{1}{8}$ (c) $-2 < m \leq \frac{1}{4}$ (d) $-6 < m < -2$

11 Find the S.S. of the inequality : $10 > X^2 + 2X - 5 \geq 3$ in \mathbb{R}



Life Applications on Unit One

From the school book



« 2 sec. or 3 sec. »

- 1 A missile is launched vertically upwards with speed $u = 24.5$ m./sec. Calculate the time "t" in seconds elapsed such that the missile reaches a height $S = 29.4$ m., given that the relation between the height "S" and the time "t" is as follows : $S = ut - 4.9t^2$

- 2 A diver starts jumping from a platform of height 10 metres above water surface. If the height of the diver above water surface "S" metres is determined by the relation : $S = -4.9t^2 + 3.5t + 10$, where "t" is the time in seconds. After how many seconds the diver will reach the water surface ?



« $\frac{5}{7}$ sec. »

- 3 The dimensions of a rectangular piece of land are 6 and 9 metres, it is required to double its area by increasing each of its dimensions with the same magnitude. Find the additional magnitude.

« 3 metres »

- 4 A golf player strikes the ball to a certain place, the following relation represents the height "y" in feet : $y = -16t^2 + 80t + 20$ where "t" is the time by sec.

- (1) After how many seconds it will reach the ground surface ?
(2) Does the ball reach a height 130 feet ?



« 5.24 sec. »

- 5 Population of Egypt in 2013 is estimated by the relation : $Z = n^2 + 1.2n + 91$, where (n) is the number of years and (Z) is the population in millions :

- (1) What is the population in 2013 ?
(2) Estimate the population in 2023
(3) Estimate the number of years at which the population will be 334 million.

« 91 million, 203 million, 15 years i.e. in 2028 »

- 6** Find the total electric current intensity passing through two resistances connected in parallel in a closed circuit, if the current intensity in the first resistance is $(4 - 2i)$ ampere and the second resistance is $\left(\frac{6+3i}{2+i}\right)$ ampere (given that the total current intensity equals the sum of the two current intensities which passes through the two resistances).

« $(7 - 2i)$ ampere »

- 7** If the electric current intensity passing in two resistances connected on parallel in a closed circuit equals $6 + 4i$ ampere, and the current intensity passing in one of them equals $\frac{17}{4-i}$, then find the current intensity passing in the other resistance.

« $(2 + 3i)$ ampere »

- 8** The production of a gold mine from 1990 to 2010 estimated in determined ounce was determined by the function $f : f(n) = 12n^2 - 96n + 480$ where 'n' is the number of years and $f(n)$ is the production of gold.

- (1) Investigate the sign of the production function f
- (2) Find the production of the gold mine (in thousand ounce) in each of the two years 1990 – 2005
- (3) In which year, the production of the gold was 2016 thousand ounce?

« 480 thousands ounces, 1740 thousands ounces, 2006 »

Unit
2

Trigonometry

Unit Exercises

- Exercise 7** : Directed angle.
Exercise 8 : Systems of measuring angle (Degree measure - radian measure).
Exercise 9 : Trigonometric functions.
Exercise 10 : Related angles.
Exercise 11 : Graphing trigonometric functions.
Exercise 12 : Finding the measure of an angle given the value of one of its trigonometric ratios.

At the end of the unit :

- Life applications on unit two.

Exercise

7

Directed angle

Test
yourself

From the school book

1 Choose the correct answer from those given :

(1) The ordered pair $(\overrightarrow{OB}, \overrightarrow{OC})$ represents the directed angle

- (a) $\angle OBC$ (b) $\angle BOC$ (c) $\angle BCO$ (d) $\angle OCB$

(2) It is said that the directed angles in the standard positions are equivalent if they have the same

- (a) initial side. (b) terminal side. (c) vertex. (d) rotation direction.

(3) If θ is the smallest positive measure of a directed angle, then its negative measure is

- (a) $-\theta$ (b) $\theta - 180^\circ$ (c) $\theta - 360^\circ$ (d) $360^\circ - \theta$

(4) If the directed angle is in standard position, which of the following is correct ?

- ① its vertex is the origin.
② its initial side coincides the positive x -axis.
③ its measure is positive.

- (a) ① only (b) ①, ② only (c) ①, ③ only (d) All the previous

(5) If θ is the directed angle measure in standard position, $n \in \mathbb{Z}$, then the angles whose measures $(\theta \pm n \times 360^\circ)$ are called

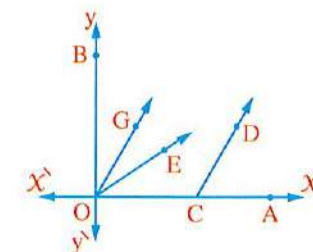
- (a) equivalent. (b) quadrantal. (c) supplementary (d) adjacent

(6) The quadrantal angle measure is multiple of angle

- (a) 360° (b) 180° (c) 90° (d) 60°

2 Which one of the following ordered pairs expresses a directed angle in its standard position ? Explain your answer.

- (1) $(\overrightarrow{CA}, \overrightarrow{CD})$ (2) $(\overrightarrow{OA}, \overrightarrow{OE})$
(3) $(\overrightarrow{OE}, \overrightarrow{OA})$ (4) $(\overrightarrow{OA}, \overrightarrow{OG})$
(5) $(\overrightarrow{OB}, \overrightarrow{OG})$ (6) $(\overrightarrow{OA}, \overrightarrow{OB})$



3 Which of the following directed angles is in its standard position ?

Explain your answer.

(1)

(2)

(3)

(4)

(5)

(6)

(7)

(8)

(9)

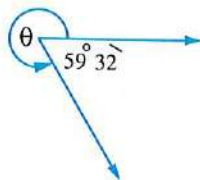
4 Find the measure of the directed angle θ in each of the following .

(1)

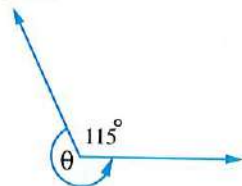
(2)

(3)

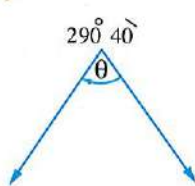
(4)



(5)



(6)



5 Show by drawing, each of the following angles in the standard position :

- (1) 32° (2) 140° (3) -80° (4) -110° (5) -315°

6 Determine the quadrant in which each of the following angles lies :

- (1) 24° (2) 215° (3) -50° (4) -210°
 (5) $150^\circ 14'$ (6) $89^\circ 59'$ (7) -180° (8) $269^\circ 59' 60''$

7 Determine the smallest positive measure for each of the angles whose measures are as follows, then determine the quadrant in which each angle lies :

- (1) -56° (2) 600° (3) -215° (4) 940°
 (5) 415° (6) -870° (7) $1120^\circ 15'$ (8) $-590^\circ 18'$

8 Determine one of the negative measures for each of the angles of the following measures :

- (1) 83° (2) 136° (3) 90°
 (4) 264° (5) 964° (6) 1070°

9 Find two angles, one of them with positive measure and the other with negative measure having common terminal side for each of the following angles :

- (1) 40° (2) 150° (3) -125° (4) -240° (5) -180°

10 Choose the right answer :

- (1) The angle whose measure is 60° in the standard position is equivalent to the angle of measure
 (a) 120° (b) 240° (c) 300° (d) 420°
 (2) The angle of measure 585° is equivalent to the angle in the standard position of measure
 (a) 45° (b) 135° (c) 225° (d) 315°

(3) The angle whose measure is 950° is equivalent to the angle in the standard position of measure

- (a) 130° (b) -130° (c) 235° (d) -230°

(4) All the following angles are equivalent to 75° in the standard position except

- (a) -285° (b) -645° (c) 285° (d) 435°

(5) The quadrant in which the angle of measure 1670° lies is the

- (a) first. (b) second. (c) third. (d) fourth.

(6) The angle whose measure is (-135°) lies in the quadrant.

- (a) first (b) second (c) third (d) fourth

(7) The angle whose measure is (-850°) lies in the quadrant.

- (a) first (b) second (c) third (d) fourth

(8) All the following are measures of angles lying in the second quadrant except

- (a) -240° (b) 100° (c) -120° (d) 860°

(9) The angle of measure $45^\circ + (4n + 1) \times 90^\circ$ lies in the quadrant ($n \in \mathbb{Z}$)

- (a) first (b) second (c) third (d) fourth

(10) If A and B are the measures of two equivalent angles, then $-A$ and $-B$ are

- (a) supplementary. (b) equivalent.
 (c) complementary. (d) of sum -360°



Discover the error

11 Write the positive measure of the smallest angle and another angle with negative measure sharing with the terminal side for the angle whose measure is (-135°) :

Karim's answer

The smallest angle with positive measure $= -135^\circ + 180^\circ = 45^\circ$
 An angle with negative measure $= -135^\circ - 180^\circ = -315^\circ$

Ziad's answer

The smallest angle with positive measure $= -135^\circ + 360^\circ = 225^\circ$
 An angle with negative measure $= -135^\circ - 360^\circ = -495^\circ$

Which of the two answers is correct ?

Problems that measure high standard levels of thinking

12 Choose the correct answer from those given :

- (1) If A, B are two measures of equivalent angles, then which of the following represents the measures of equivalent angles, where $C \in \mathbb{Z}$?
- (a) $(A + C), (B + C)$ (b) $(A - C), (B - C)$
 (c) $(CA), (CB)$ (d) All the previous
- (2) If $A, -A$ are measures of two equivalent angles, then one of the values of A is
- (a) 150° (b) 90° (c) 180° (d) 270°
- (3) If $(3x - 5)^\circ$ is the smallest positive measure, $(3y - 5)^\circ$ is the greatest negative measure of equivalent angles, then $x - y = \dots\dots\dots$
- (a) 360° (b) 180° (c) 120° (d) 90°
- (4) If $(\theta + 20)^\circ, (20 - 8\theta)^\circ$ are the positive and negative measures of a directed angle respectively, then the smallest positive value of θ is
- (a) 20° (b) 10° (c) 30° (d) 40°
- (5) If the terminal side of angle 60° in standard position rotates two and quarter revolutions anticlockwise then the terminal side represents the angle
- (a) 60° (b) 120° (c) 150° (d) 240°
- (6) If the terminal side of an angle of measure 30° in standard position rotates three and half revolutions clockwise, then the terminal side will be in the quadrant.
- (a) first (b) second (c) third (d) fourth
- (7) If the terminal side of an angle in standard position passes through the point $(-1, 0)$, then its terminal side lies in
- (a) first quadrant. (b) second quadrant.
 (c) third quadrant. (d) otherwise.

Exercise

8

Systems of measuring angle (Degree measure - Radian measure)

From the school book

Test
yourself

1 Find in terms of π the radian measure of each of the angles whose measures are as follows :

- | | | | |
|------------------|---------------------|-----------------|------------------|
| (1) 135° | (2) 90° | (3) 300° | (4) -235° |
| (5) -210° | (6) $112^\circ 30'$ | (7) 390° | (8) 780° |

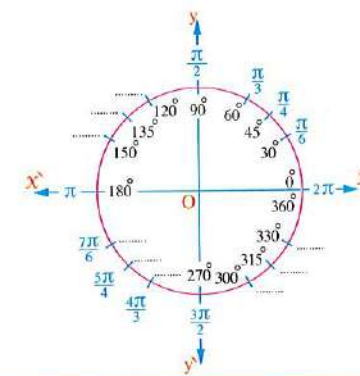
2 Find the radian measure of each of the angles whose degree measures are as follows approximating the result to three decimal places :

- | | | |
|-------------------------|---------------------|--------------------------|
| (1) 58° | (2) 56.6° | (3) $37^\circ 15'$ |
| (4) $115^\circ 38' 6''$ | (5) $257^\circ 54'$ | (6) $160^\circ 50' 48''$ |

3 Find the degree measure (in degrees, minutes and seconds) of each of the angles whose radian measures are as follows :

- | | | |
|--------------------------|-------------------------|----------------------------------|
| (1) $\frac{11\pi}{15}$ | (2) 0.72π | (3) 0.49^{rad} |
| (4) -1.67^{rad} | (5) 2.27^{rad} | (6) $-3\frac{1}{2}^{\text{rad}}$ |

4 The opposite figure represents the measures of some special angles, some of them is written in radian outside the circle, and the other is written in degrees inside the circle. Write the corresponding measure of each angle in the opposite figure.



- 5** Determine the degree measure and the radian measure for the central angle that subtends an arc of length (l) in a circle of radius (r) in each of the following cases :

(1) $l = 12$ cm. , $r = 10$ cm.	(2) $l = 14$ cm. , $r = 7$ cm.
(3) $l = 2\pi$ cm. , $r = 6$ cm.	(4) $l = 15.72$ cm. , $r = 9.17$ cm.

- 6** Find the length of the radius of the circle in which a central angle (θ) is drawn subtending an arc of length (l) in each of the following cases :

(1) $\theta = \frac{9\pi}{8}$, $l = 22.5$ cm.	(2) $\theta = 0.767^{\text{rad}}$, $l = 38.35$ cm.
(3) $\theta = 139^\circ$, $l = 24.325$ cm.	(4) $\theta = 78^\circ 36' 26''$, $l = 43.92$ cm.

- 7** Find to the nearest one decimal place of a centimetre the length of an arc in a circle of radius length (r) subtending a central angle of measure (θ) in each of the following cases :

(1) $r = 12.5$ cm. , $\theta = 1.6^{\text{rad}}$	(2) $r = 20$ cm. , $\theta = 2.43^{\text{rad}}$
(3) $r = 7.5$ cm. , $\theta = 67^\circ 40'$	(4) $r = 15$ cm. , $\theta = 104^\circ 58' 6''$

- 8** Choose the right answer :

- (1) The angle of measure $\frac{25\pi}{9}$ lies in the quadrant.
 (a) first (b) second (c) third (d) fourth
- (2) The angle of measure $\frac{31\pi}{6}$ lies in the quadrant.
 (a) first (b) second (c) third (d) fourth
- (3) The angle of measure $\frac{-9\pi}{4}$ lies in the quadrant.
 (a) first (b) second (c) third (d) fourth
- (4) If the degree measure of an angle is $43^\circ 12'$, then its radian measure is
 (a) 0.24^{rad} (b) 0.24π (c) 0.28^{rad} (d) 0.28π
- (5) The degree measure of the angle of measure $\frac{8\pi}{3}$ is
 (a) 540° (b) 820° (c) 150° (d) 480°
- (6) The sum of the measures of the angles of the quadrilateral in radian equals
 (a) 2π (b) π (c) $\frac{3\pi}{2}$ (d) 3π
- (7) If the sum of measures of the interior angles of a regular polygon equals $180^\circ (n - 2)$ where n is the number of its sides , then the measure of the interior angle in radian of a regular pentagon equals
 (a) $\frac{\pi}{3}$ (b) $\frac{7\pi}{2}$ (c) $\frac{3\pi}{5}$ (d) $\frac{2\pi}{3}$
- (8) In a circle of diameter length 12 cm. , the length of the arc subtended by a central angle of measure 60° equals cm.
 (a) 5π (b) 4π (c) 3π (d) 2π

- (9) The measure of the central angle in a circle of radius length 15 cm. and opposite to an arc of length 5π cm. equals
 (a) 30° (b) 60° (c) 90° (d) 180°

- (10) If the measure of one of the angles of a triangle is 75° and the measure of another angle is $\frac{\pi}{3}$, then the radian measure of the third angle equals
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{12}$

- (11) The string length of a simple pendulum is 14 cm. swings in an angle of measure $\frac{1}{10}\pi$, then its arcs length \approx cm.
 (a) 4.6 (b) 4.4 (c) 4.2 (d) 4.8

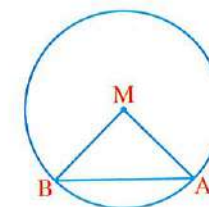
- (12) ABCD is a cyclic quadrilateral , $m(\angle A) = 60^\circ$, then $m(\angle C) =$
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

- (13) In the opposite figure :

To find the length of \widehat{AB}

its sufficient to get

- (a) $\triangle AMB$ is an equilateral triangle of perimeter 30 cm. only.
 (b) The circle circumference = 10π cm only.
 (c) (a) , (b) together.
 (d) nothing of the previous.



- (14) The radian measure of a regular heptagon exterior angle equals
 (a) $\frac{1}{7}\pi$ (b) $\frac{2}{7}\pi$ (c) $\frac{3}{7}\pi$ (d) $\frac{4}{7}\pi$

- (15) In the opposite figure :

If \overline{AB} , \overline{AC} are two tangent

to the circle M and $m(\angle A) = \frac{5}{12}\pi$

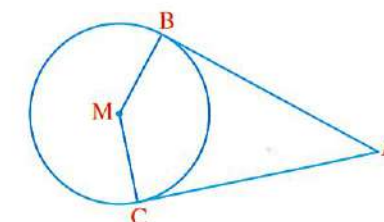
and the circle circumference = 96 cm.

, then the smaller arc length $\widehat{BC} =$

- (a) 20 (b) $\frac{28}{\pi}$ (c) 28 (d) 20π

- (16) The angle whose measure $30^\circ + 180^\circ (2n + 1)$ where $n \in \mathbb{Z}$, its radian measure is equivalent to
 (a) $\frac{\pi}{6}$ (b) π (c) $\frac{7}{6}\pi$ (d) $\frac{5}{3}\pi$

- (17) If the length of an arc in a circle equals $\frac{3}{8}$ of its circumference , then the measure of the central angle subtending this arc in degrees equals
 (a) 30° (b) $67^\circ 30'$ (c) 135° (d) 43° approximately.



(18) In the circle whose radius length is the unit length, the measure of the central angle in radian is

- (a) $\frac{1}{4}$ its arc length. (b) $\frac{1}{2}$ its arc length.
(c) the length of the arc. (d) double its arc length.

9 Find the circumference of a circle which has an arc of length 12 cm. subtended by an inscribed angle of measure 45° « 48 cm. »

10 Find in radian and degrees the measure of a central angle subtending an arc of length three times the length of the radius of its circle. « 3^{rad} , $171^\circ 53' 14''$ »

11 If the measure of a central angle in a circle equals 105° and it is subtending an arc of length $\frac{7\pi}{3}$ cm., find the length of the diameter of the circle. « 8 cm. »

12 In a triangle, the measure of one of its angles is 60° , and the measure of another angle is $\frac{\pi}{4}$. Find the radian measure and the degree measure of the third angle. « $\frac{5}{12}\pi$, 75° »

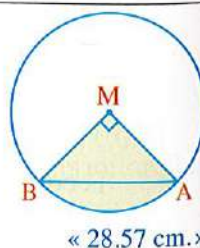
13 In a quadrilateral, the measure of one of its angles is $\frac{11}{6}^{\text{rad}}$, the measure of another angle is $2\frac{4}{9}^{\text{rad}}$ and the measure of a third angle is 45° . Find the degree measure and the radian measure of the fourth angle ($\pi \approx \frac{22}{7}$) « 70° , $\frac{11}{9}^{\text{rad}}$ »

14 Two angles, the sum of their measures equals 70° , and the difference between them equals $\frac{\pi}{5}$, find the measure of each angle in degrees and in radian. « 53° , 17° , $\frac{53}{180}\pi$, $\frac{17}{180}\pi$ »

15 Two supplementary angles, the difference between their measures is $\frac{\pi}{3}$. Find the measures of the two angles in radian and in degrees. « $\frac{2\pi}{3}$, $\frac{\pi}{3}$, 120° , 60° »

16 In the opposite figure :

If the area of the right-angled triangle MAB at M = 32 cm^2 , find the perimeter of the shaded area to the nearest hundredth.



« 28.57 cm. »

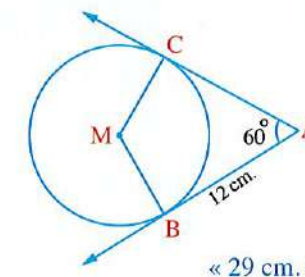
17 \overline{XY} is a diameter in circle M its length is 18 cm., the chord \overline{YZ} is drawn such that $m(\angle XYZ) = 10^\circ$. Determine the length of the minor arc \widehat{XZ} approximating the result to the nearest two decimal places. « 3.14 cm. »

18 In the opposite figure :

\overline{AB} , \overline{AC} are two tangents to the circle M,

$m(\angle CAB) = 60^\circ$, $AB = 12 \text{ cm}$.

Find to the nearest integer the length of the greater arc \widehat{BC}



« 29 cm. »

19 ABC is a right-angled triangle at C drawn inside a circle, if $AB = 24 \text{ cm}$, $BC = 12 \text{ cm}$, find the lengths of the three arcs into which the circle is divided by the vertices of this triangle approximating the result to the nearest one decimal place. « 12.6 cm., 25.1 cm., 37.7 cm. »

20 A circle of radius length 7.5 cm. passing through the vertices of the triangle ABC, if $m(\angle BAC) = 60^\circ$, $m(\angle ABC) = 54^\circ$, find the lengths of the three arcs into which the circle is divided by the vertices of this triangle. « 15.7 cm., 14.1 cm., 17.3 cm. »

Problems that measure high standard levels of thinking

21 Choose the correct answer from those given :

(1) If an arc opposite to central angle of measure 72° was cut from a circle whose radius length 14 cm. and bent to form a circle, then the radius length of the resulted circle = cm.

- (a) 1.4 (b) 2.8 (c) 5.6 (d) 7

(2) In the opposite figure :

Circle whose centre M, the radius length 10 cm., if the length of $\widehat{AB} \in]5, 6[$, then the value of X could be

- (a) 90° (b) 60° (c) 28° (d) 34°

(3) If the ratio between measures of angles of a quadrilateral is 5 : 4 : 9 : 6, then the measure of the smallest angle =^{rad}

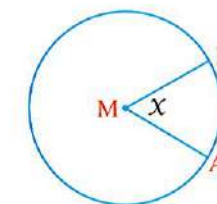
- (a) $\frac{\pi}{12}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{12}$ (d) $\frac{3\pi}{4}$

(4) The positive measure of an angle that formed between the hour hand and the minute hand at exactly half past two equals^{rad}

- (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{12}$ (c) $\frac{7\pi}{12}$ (d) $\frac{3\pi}{4}$

(5) If the arc length opposite to central angle of measure 60° in a circle equals the arc length opposite to central angle of measure 80° in another circle, then the ratio between the two radii of the two circles is

- (a) $\frac{5}{4}$ (b) $\frac{4}{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{9}{16}$



(6) A cylinder rotates 45 revolutions per minute around its axis, then the measure of the angle at which a point on the lateral surface rotates in one second equals

- (a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

(7) (The measure of the circle)^{rad} > n where n is a positive integer, then the greatest value for n is

- (a) 3 (b) 5 (c) 6 (d) 8

(8) The distance covered by the tip of the minute hand whose length 8 cm. from 6 am till quarter past three pm equals cm.

- (a) 592π (b) 148π (c) $\frac{37}{2}\pi$ (d) $\frac{37}{4}\pi$

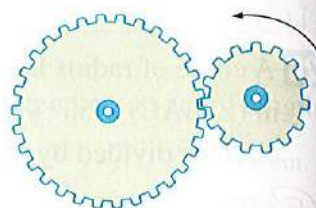
(9) In the opposite figure :

When the greater gear revolves one revolution then the smaller gear revolves 3 revolutions.

If the smaller gear revolves one revolution in the direction of the arrow shown on the figure

, then the measure of the central angle of revolving the greater gear is rad

- (a) $-\frac{\pi}{2}$ (b) $-\frac{2\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) 2π

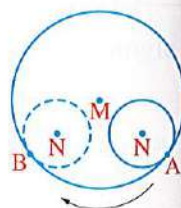


(10) In the opposite figure :

Two circles M and N, their radii length are 21 cm.,

7 cm. respectively. If a circle N rotated a complete revolution from a point A to point B, then $m(\angle AMB) = \dots\dots\dots$

- (a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{2\pi}{5}$ (d) π

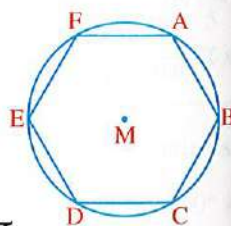


(11) In the opposite figure :

ABCDEF is a regular hexagon of side length 4 cm. inscribed in a circle M

, then the length of $\widehat{AB} = \dots\dots\dots$ cm.

- (a) π (b) $\frac{4}{3}\pi$ (c) 2π (d) $\frac{5}{3}\pi$



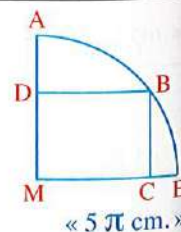
22 A straight line makes an angle of radian measure $\frac{\pi}{3}$ with the positive direction of the X-axis in the standard position in the unit circle. Find the equation of the straight line.

« $y = \sqrt{3}x$ »

23 In the opposite figure :

A quarter circle, BCMD is a rectangle which is drawn inside it, where $CD = 10$ cm.

Find the length of arc : \widehat{ABE}



« 5π cm. »



Trigonometric functions

From the school book



Test
yourself

1 Choose the correct answer from those given :

(1) If θ is the measure of an angle in the standard position, its terminal side intersects the unit circle at the point $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, then $\sin \theta = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{3}}$

(2) If $\sin \theta = \frac{1}{\sqrt{2}}$, where θ is the measure of a positive acute angle, then the measure of angle $\theta = \dots\dots\dots$

- (a) 30° (b) 60° (c) 45° (d) 90°

(3) If $\sin \theta = -1$, $\cos \theta = 0$, then the measure of angle $\theta = \dots\dots\dots$

- (a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

(4) If $\csc \theta = 2$, where θ is a positive acute angle, then the measure of angle $\theta = \dots\dots\dots$

- (a) 15° (b) 30° (c) 45° (d) 60°

(5) If $\tan \theta = 1$, where θ is a positive acute angle, then the measure of angle $\theta = \dots\dots\dots$

- (a) 60° (b) 30° (c) 45° (d) 90°

(6) If $\cos \theta = \frac{\sqrt{3}}{2}$, where θ is a positive acute angle, then $\sin \theta = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

(7) $\cot^2 30^\circ - \sec^2 60^\circ + \csc^2 45^\circ = \dots\dots\dots$

- (a) 1 (b) 0 (c) -1 (d) 2

(8) If $\sin \theta = \frac{-1}{2}$, $\cos \theta = \frac{\sqrt{3}}{2}$, then angle of measure θ lies in the $\dots\dots\dots$ quadrant.

- (a) first (b) second (c) third (d) fourth

(9) $\sin\left(-\frac{12}{5}\pi\right) = \dots\dots\dots$

- (a) $\sin \frac{12}{5}\pi$ (b) $\sin 72^\circ$ (c) $\sin 288^\circ$ (d) $\sin \frac{1}{5}\pi$

(10) $\sin 0^\circ + \cos 0^\circ + \tan 0^\circ = \dots\dots\dots$

- (a) 0 (b) 1 (c) 2 (d) 3

(11) $2 \sin 45^\circ = \dots\dots\dots$

- (a) $\sin 90^\circ$ (b) $\frac{\sqrt{2}}{2}$ (c) $\sqrt{2}$ (d) 2

(12) If $\cos \theta = \frac{1}{2}$, $\sin \theta = -\frac{\sqrt{3}}{2}$, then the measure of angle $\theta = \dots\dots\dots$

- (a) $\frac{2\pi}{3}$ (b) $\frac{5\pi}{6}$ (c) $\frac{5\pi}{3}$ (d) $\frac{11\pi}{6}$

(13) If the terminal side of an angle in standard position intersects the unit circle of point A which lies in the fourth quadrant where the X-coordinate of A equals $\frac{5}{13}$, then A = $\dots\dots\dots$

- (a) $\left(\frac{5}{13}, -\frac{12}{13}\right)$ (b) $\left(\frac{5}{13}, \frac{1}{13}\right)$ (c) $\left(\frac{5}{13}, \frac{12}{13}\right)$ (d) $\left(\frac{5}{13}, -\frac{8}{13}\right)$

(14) If ABC is a right angled triangle at B, $m(\angle A) = 2m(\angle C)$, then $\sec A + \csc C = \dots\dots\dots$

- (a) 2 (b) 4 (c) 6 (d) 8

2 Determine the signs of the following trigonometric functions :

(1) $\cos 350^\circ$

(2) $\tan 100^\circ$

(3) $\sec 265^\circ$

(4) $\sin \frac{5\pi}{4}$

(5) $\csc \frac{3\pi}{7}$

(6) $\cot \frac{3\pi}{4}$

(7) $\tan 410^\circ$

(8) $\csc 1200^\circ$

(9) $\cos(-165^\circ)$

(10) $\cot \frac{32\pi}{3}$

(11) $\cot\left(-\frac{3\pi}{4}\right)$

(12) $\sec\left(-\frac{25\pi}{6}\right)$

3 Find all trigonometric functions of the angle whose measure is θ drawn in the standard position, its terminal side intersects the unit circle at the point :

(1) $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

(2) $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

(3) $(0, -1)$

4 If θ is the measure of a directed angle in the standard position and B is the intersection point of its terminal side with the unit circle, then find all trigonometric functions of the angle θ in each of the following cases :

(1) B $(0.6, y)$, $y > 0$

(2) B $(x, -0.6)$, $x > 0$

(3) B $\left(-\frac{\sqrt{3}}{2}, y\right)$, where $90^\circ < \theta < 180^\circ$

(4) B $\left(x, \frac{\sqrt{5}}{3}\right)$, $x < 0$

(5) B $(-1, y)$

(6) B $(-x, x)$, $x > 0$

(7) B $(-x, -x)$, $x > 0$

(8) B $(9a, 12a)$ where $180^\circ < \theta < 270^\circ$

(9) B $\left(\frac{3}{2}a, -2a\right)$, where $\frac{3\pi}{2} < \theta < 2\pi$

5 Find the value of each of :

(1) $\sin 0^\circ + \sin 90^\circ + \sin 180^\circ + \sin 270^\circ$

(2) $\tan 0^\circ + \tan 45^\circ + \tan 180^\circ$

(3) $\sin 30^\circ + \cos 60^\circ - \cot 45^\circ$

(4) $\tan^2 30^\circ + 2 \sin^2 45^\circ + \cos^2 90^\circ$

(5) $\sin 180^\circ \cos 45^\circ - \cos 180^\circ \sin 45^\circ$

(6) $\sec \frac{\pi}{6} \tan \frac{\pi}{3} - \cot \frac{\pi}{3} \cos \frac{\pi}{6}$

(7) $\cos \frac{\pi}{2} \cos 0^\circ + \sin \frac{3\pi}{2} \sin \frac{\pi}{2}$

(8) $\frac{\tan^2 60^\circ - \tan^2 45^\circ}{\sec^2 30^\circ - \csc^2 45^\circ}$

(9) $\frac{4 \sin^2 30^\circ - 3 \tan 45^\circ \cos 0^\circ}{2 \cos 60^\circ + 2 \sin 45^\circ \cos 45^\circ}$

(10) $3 \sin 30^\circ \sin^2 60^\circ - \cos 0^\circ \sec 60^\circ + \sin 270^\circ \cos^2 45^\circ$

6 Prove each of the following equalities :

(1) $2 \sin^2 90^\circ = -2 \cos 180^\circ$

(2) $\cos \frac{\pi}{2} = \cos^2 \frac{\pi}{4} - \sin^2 \frac{\pi}{4}$

(3) $2 \sin 45^\circ \cos 45^\circ \cot 45^\circ = 1$

(4) $3 \cos 30^\circ \tan 60^\circ - 2 \sec 45^\circ \csc 45^\circ = \frac{1}{2}$

(5) $\cos^2 30^\circ \cot^2 60^\circ \tan 45^\circ = \frac{1}{4}$

(6) $3 \cot^2 45^\circ - 2 \sin 60^\circ \cos 30^\circ = \frac{3}{2} \sin^2 90^\circ$

(7) $\sec 30^\circ \tan 60^\circ + \csc^2 60^\circ - \tan^2 45^\circ = \frac{7}{3}$

(8) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ = \sin^2 \frac{\pi}{4}$

(9) $3 \tan^2 30^\circ + \frac{4}{3} \cos^2 30^\circ - \frac{1}{4} \cot^2 45^\circ \csc^2 30^\circ = 1$

$$(10) 2 \cos^2 \frac{\pi}{3} + 3 \sin^2 \frac{\pi}{4} + 4 \tan^2 \frac{\pi}{3} - 4 \sin \frac{\pi}{2} = 10$$

$$(11) \frac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ} = \cot 60^\circ$$

$$(12) \frac{\sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ}{\sin 45^\circ \cos 60^\circ + \cos 45^\circ \sin 60^\circ} = \sin 90^\circ$$

7 Find the value of X if :

$$(1) X \sin^2 \frac{\pi}{4} \cos \pi = \tan^2 \frac{\pi}{3} \sin \frac{3\pi}{2} \quad \ll 6 \gg$$

$$(2) X \sin \frac{\pi}{4} \cos \frac{\pi}{4} \cot \frac{\pi}{6} = \tan^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{3} \quad \ll \frac{\sqrt{3}}{2} \gg$$

8 If $X \in [0^\circ, 90^\circ]$, then find the value of X which satisfies each of the following equations :

$$(1) \cos X = \frac{\sin 60^\circ}{\sin 90^\circ} - \frac{\sin 0^\circ}{\sin 45^\circ} \quad \ll 30^\circ \gg$$

$$(2) \sin X = \sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ \quad \ll 90^\circ \gg$$

9 Find all trigonometric ratios for the angle AOB whose measure is θ in each of the following cases :

$$(1) \theta \in]0, \frac{\pi}{2}[, \cos \theta = 0.6 \quad (2) \theta \in]\frac{\pi}{2}, \pi[, \sin \theta = \frac{12}{13}$$

$$(3) \theta \in]\frac{\pi}{2}, \pi[, \tan \theta = -\frac{3}{4} \quad (4) \theta \in]\pi, \frac{3\pi}{2}[, \csc \theta = -\frac{25}{7}$$

$$(5) \theta \in]\frac{3\pi}{2}, 2\pi[, \sec \theta = 2$$

10 If the terminal side of the angle θ in the standard position intersects the unit circle at the point $(2a, 3a)$, where $0 < \theta < \frac{\pi}{2}$, find the value of a , then find the value of :

$$\sec^2 \theta - \tan^2 \theta \quad \ll \frac{1}{\sqrt{13}}, 1 \gg$$

11 If $\theta \in]0, \frac{\pi}{2}[, \cos \theta = \frac{3}{5}$, then find the value of : $\csc \theta \sin \theta - \tan \theta \csc \theta \quad \ll -\frac{2}{3} \gg$

12 If $\theta \in]\frac{\pi}{2}, \pi[, \sin \theta = \frac{12}{13}$, then find the value of : $\csc \theta \sin \theta - \tan \theta \cot \theta + \cos^2 \theta \quad \ll \frac{25}{169} \gg$

13 If $\theta \in]\frac{3\pi}{2}, 2\pi[, \sin \theta = -\frac{24}{25}$, then find :

$$(1) \frac{\cot \theta - \csc \theta}{\tan \theta - \sec \theta}$$

$$(2) \cos \theta - \csc \theta \tan \theta \quad \ll -\frac{3}{28}, -\frac{576}{175} \gg$$



Discover the error

14 The teacher asks the students to find the value of : $2 \sin 45^\circ$

Karim's answer

$$2 \sin 45^\circ = \sin 2 \times 45^\circ \\ = \sin 90^\circ = 1$$

Ahmed's answer

$$2 \sin 45^\circ = 2 \times \frac{1}{\sqrt{2}} \\ = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

Which of the two answers is correct ? Why ?



Problems that measure high standard levels of thinking

15 Choose the correct answer from those givens :

(1) In the unit circle whose centre is (O) if the length of $\widehat{BC} = \frac{1}{3} \pi$, then $\sec (\angle BOC) = \dots\dots\dots$

$$(a) \frac{\sqrt{3}}{2} \quad (b) \frac{1}{2} \quad (c) \frac{-1}{2} \quad (d) 2$$

(2) If A is the greatest acute angle measure in a triangle whose side lengths are 5, 12, 13 cm., then $\cot A = \dots\dots\dots$

$$(a) \frac{12}{13} \quad (b) \frac{5}{13} \quad (c) \frac{5}{12} \quad (d) \frac{12}{5}$$

(3) If the side lengths of right-angled triangle ABC are $X-7$, X , $X+1$ and \widehat{BC} is the smallest side, then $\sec A = \dots\dots\dots$

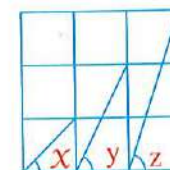
$$(a) \frac{5}{13} \quad (b) \frac{12}{13} \quad (c) \frac{13}{12} \quad (d) \frac{5}{4}$$

(4) In the opposite figure :

All squares are identical

, then $\cot X + \cot y + \cot z = \dots\dots\dots$

$$(a) 6 \quad (b) \frac{11}{6} \quad (c) \frac{6}{11} \quad (d) \sqrt{5} + 3$$

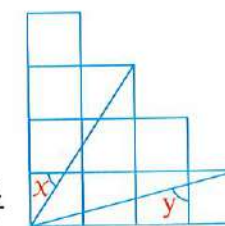


(5) In the opposite figure :

All squares are identical

, then $\tan X + \cot y = \dots\dots\dots$

$$(a) \frac{11}{12} \quad (b) \frac{7}{4} \quad (c) \frac{5}{3} \quad (d) \frac{14}{3}$$

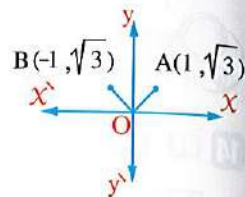


(6) In the opposite figure :

If $A(1, \sqrt{3})$, $B(-1, \sqrt{3})$

, then $\cot(\angle AOB) = \dots\dots\dots$

- (a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$



(7) In the opposite figure :

O is the centre of unit circle \overline{AB}

is a tangent segment , then :

First : $OB = \dots\dots\dots$

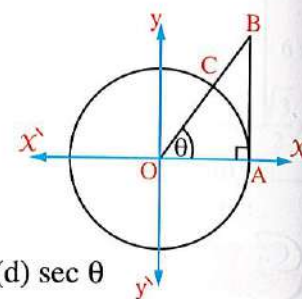
- (a) $\sin \theta$ (b) $\cos \theta$ (c) $\csc \theta$

second : $BC = \dots\dots\dots$

- (a) $\cot \theta$ (b) $(\sec \theta) - 1$ (c) $(\csc \theta) - 1$ (d) $\cos \theta$

Third : The area of triangle ABO = $\dots\dots\dots$

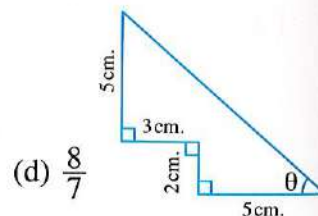
- (a) $\frac{1}{2} \cos \theta$ (b) $\frac{1}{2} \tan \theta$ (c) $\frac{1}{2} \sin \theta$ (d) $\frac{1}{2} \sin \theta \cos \theta$



(8) In the opposite figure :

$\cot \theta = \dots\dots\dots$

- (a) $\frac{2}{5}$ (b) $\frac{7}{8}$ (c) $\frac{3}{2}$

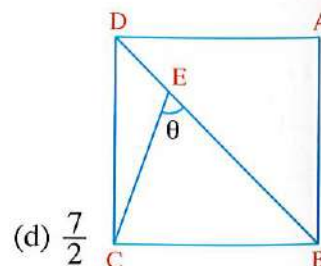


(9) In the opposite figure :

If ABCD is a square and $\frac{DE}{EB} = \frac{2}{5}$

, then $\tan \theta = \dots\dots\dots$

- (a) $\frac{7}{3}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$

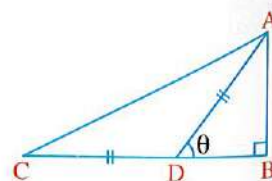


(10) In the opposite figure :

If $D \in \overline{BC}$ and $AD = DC$

, $\tan \theta = \frac{4}{3}$, then $\cot \frac{\theta}{2} = \dots\dots\dots$

- (a) $\frac{3}{4}$ (b) 2 (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

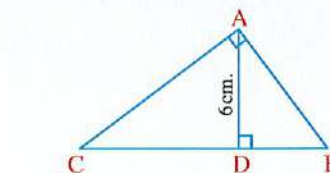


(11) In the opposite figure :

If $\tan B + \tan C = \frac{5}{3}$

, then $BC = \dots\dots\dots$ cm.

- (a) 6 (b) 8 (c) 10

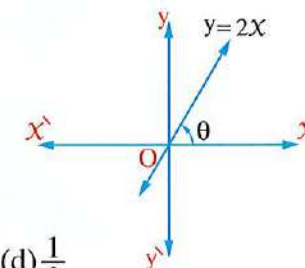


(d) 14

(12) In the opposite figure :

If θ is the included angle between the straight line $y = 2x$ and the positive direction of x -axis , then $\sin \theta = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{1}{3}$



Exercise

10

Related angles



Test

From the school book

yourself

1 Choose the correct answer :

(1) $\tan 42^\circ = \dots\dots\dots$

- (a) $\cot 42^\circ$ (b) $\tan 48^\circ$ (c) $\cot 48^\circ$ (d) $\csc 48^\circ$

(2) $\cot (90^\circ + \theta) = \dots\dots\dots$

- (a) $\tan (90^\circ - \theta)$ (b) $-\tan \theta$ (c) $\tan (90^\circ + \theta)$ (d) $\tan (270^\circ + \theta)$

(3) $\frac{\sec 105^\circ}{\csc 15^\circ} = \dots\dots\dots$

- (a) $\frac{\sin 105^\circ}{\cos 15^\circ}$ (b) $\tan 135^\circ$ (c) $\cot 15^\circ$ (d) $\cos 90^\circ$

(4) $\cos \theta + \cos (180^\circ - \theta) = \dots\dots\dots$

- (a) zero (b) 1 (c) $2 \cos \theta$ (d) $\cos \theta$

(5) $\sin \theta + \cos (270^\circ + \theta) = \dots\dots\dots$

- (a) zero (b) 1 (c) $2 \sin \theta$ (d) $\sin \theta \cos \theta$

(6) If ABC is an acute angled triangle, then $\cos A + \cos (B + C) = \dots\dots\dots$

- (a) -1 (b) zero (c) 1 (d) $\frac{1}{2}$

(7) If θ is the measure of an angle in standard position and its terminal side intersects the unit circle at $(X, -X)$ where $X > 0$, then $\theta = \dots\dots\dots^\circ$

- (a) 45 (b) 135 (c) 225 (d) 315

(8) If $\sin \theta = -\frac{1}{2}$, θ is the smallest positive measure, then $\theta = \dots\dots\dots^\circ$

- (a) 30 (b) 150 (c) 210 (d) 330

(9) If the terminal side of an angle whose measure is θ in standard position intersects the unit circle at the point $(-\frac{\sqrt{3}}{2}, y)$ where $y \in \mathbb{R}^+$, then $\theta = \dots\dots\dots$

- (a) 30° (b) 150° (c) 210° (d) 330°

(10) If ABCD is a cyclic quadrilateral and $\sin A = \frac{3}{5}$, then $\sin C = \dots\dots\dots$

- (a) $\frac{3}{5}$ (b) $-\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $-\frac{4}{5}$

(11) $\frac{\tan (45^\circ + X)}{\cot (45^\circ - X)} = \dots\dots\dots$

- (a) -1 (b) 1 (c) $\tan (90^\circ + X)$ (d) $\cot (90^\circ + X)$

(12) If $X + y = \frac{\pi}{2}$, then $\frac{\sin X - \sin y}{\cos X - \cos y} = \dots\dots\dots$

- (a) -1 (b) zero (c) 1 (d) 2

(13) If $A + B = 90^\circ$, $\tan A = \frac{1}{3}$, then $\tan B = \dots\dots\dots$

- (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) 3

(14) If $\sin \theta = \cos 2\theta$, $\theta \in]0, \frac{\pi}{2}[$, then $\sin 3\theta = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) 1 (c) zero (d) $\frac{\sqrt{3}}{2}$

(15) If $\tan \theta = \cot 2\theta$, $0^\circ < \theta < 90^\circ$, then $\sin \theta + \cos 2\theta = \dots\dots\dots$

- (a) 1 (b) -1 (c) 2 (d) $\frac{1}{4}$

(16) If $\sin \alpha = \cos \beta$ where α and β are two acute angles, then $\tan (\alpha + \beta) = \dots\dots\dots$

- (a) $\frac{1}{\sqrt{3}}$ (b) 1 (c) $\sqrt{3}$ (d) undefined.

(17) If $\sin 2\theta = \cos 4\theta$ where θ is a positive acute angle, then $\tan (90^\circ - 3\theta) = \dots\dots\dots$

- (a) -1 (b) $\frac{1}{\sqrt{3}}$ (c) 1 (d) $\sqrt{3}$

(18) If $5 \cos (90^\circ - \theta) = 4$, $0^\circ < \theta < 90^\circ$, then $\sin \theta = \dots\dots\dots$

- (a) $\frac{5}{4}$ (b) $-\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $\frac{3}{5}$

(19) If $\cot (90^\circ + \theta) + 1 = 0$ where $0^\circ < \theta < 90^\circ$, then $\cos 4\theta = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) 1 (c) zero (d) -1

(20) If $\cos (90^\circ + \theta) + \sin (90^\circ - 2\theta) = 0$, where $\theta \in]0, \frac{\pi}{4}[$, then $\sin 2\theta = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) 1 (c) zero (d) $\frac{\sqrt{3}}{2}$

(21) If $\cos (270^\circ - \theta) = \frac{1}{2}$ where θ is the measure of the smallest positive angle, then $\theta = \dots\dots\dots$

- (a) 30° (b) 150° (c) 210° (d) 330°

(22) If $\tan \theta = -\frac{5}{12}$, $\cos \theta < 0$, then $\csc \theta = \dots\dots\dots$

- (a) $\frac{5}{13}$ (b) $-\frac{5}{13}$ (c) $\frac{13}{5}$ (d) $-\frac{13}{5}$

(23) If $\sin \theta = -\frac{1}{2}$, $\tan \theta > 0$, then $\theta = \dots\dots\dots$

- (a) 30° (b) 150° (c) 210° (d) 330°

2 Find the value of each of the following :

- | | | | |
|---|--|--------------------------|--|
| (1) $\sin 150^\circ$ | (2) $\sec 210^\circ$ | (3) $\tan 240^\circ$ | (4) $\cos (-150^\circ)$ |
| (5) $\tan 225^\circ$ | (6) $\csc \frac{11\pi}{6}$ | (7) $\cot 780^\circ$ | (8) $\cos (-900^\circ)$ |
| (9) $\sin \left(-\frac{4\pi}{3}\right)$ | (10) $\sec \left(-\frac{2\pi}{3}\right)$ | (11) $\sec (-480^\circ)$ | (12) $\sin \left(-\frac{7\pi}{4}\right)$ |

3 Find the value of each of the following :

- (1) $\cos 120^\circ + \tan 225^\circ + \csc 330^\circ + \cos 420^\circ$ « -1 »
- (2) $\sin 390^\circ \cos (-60^\circ) + \cos 30^\circ \sin 120^\circ$ « 1 »
- (3) $\sin 150^\circ \cos (-300^\circ) + \cos (930^\circ) \cot 240^\circ$ « $-\frac{1}{4}$ »
- (4) $\tan \frac{2\pi}{3} \sec \frac{11\pi}{3} + \cot \frac{11\pi}{6} \csc \frac{19\pi}{6} + \tan \frac{25\pi}{6} \csc \left(-\frac{19\pi}{3}\right)$ « $-\frac{2}{3}$ »

4 Prove each of the following equalities :

- (1) $\cos (-300^\circ) \sin 420^\circ - \cos 750^\circ \cos 660^\circ = \text{zero}$
- (2) $\sin 600^\circ \cos (-30^\circ) + \sin 150^\circ \cos (-240^\circ) = -1$
- (3) $\sin 480^\circ \cos (-60^\circ) + \cos 300^\circ \sin (-120^\circ) = \text{zero}$
- (4) $\sin 150^\circ \tan 225^\circ + \cos 315^\circ \sec (-120^\circ) + \sin (-135^\circ) \csc 210^\circ = \frac{1}{2}$

5 If the terminal side of an angle of measure θ in its standard position intersects the unit circle at the point $\left(-\frac{3}{5}, \frac{4}{5}\right)$, find :

- | | | |
|---|--|---------------------------------|
| (1) $\sin (180^\circ + \theta)$ | (2) $\cos \left(\frac{\pi}{2} - \theta\right)$ | (3) $\tan (360^\circ - \theta)$ |
| (4) $\csc \left(\frac{3\pi}{2} - \theta\right)$ | (5) $\sec (\theta + \pi)$ | (6) $\sin (\theta - \pi)$ |

6 If the directed angle of measure θ in the standard position, its terminal side passes by the point $\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$, find the following trigonometric functions :

- | | | |
|--|---------------------------------|--|
| (1) $\sin (270^\circ + \theta)$ | (2) $\sec (270^\circ + \theta)$ | (3) $\csc \left(\theta + \frac{\pi}{2}\right)$ |
| (4) $\tan \left(\frac{\pi}{2} - \theta\right)$ | (5) $\cot (\theta - 180^\circ)$ | (6) $\sec (-\theta)$ |

7 If θ is the measure of a positive acute angle in the standard position and its terminal side intersects the unit circle at the point $B\left(x, \frac{3}{5}\right)$, find the value of :

$$\sin (90^\circ - \theta) + \tan (90^\circ - \theta) \cos (90^\circ + \theta)$$

« zero »

8 If $\sin \theta = \frac{3}{5}$ where $90^\circ < \theta < 180^\circ$, find the value of :

- | | | |
|---------------------------------|---------------------------------|---------------------------------|
| (1) $\cos (180^\circ - \theta)$ | (2) $\tan (180^\circ + \theta)$ | (3) $\csc (-\theta)$ |
| (4) $\cot (360^\circ - \theta)$ | (5) $\sin (90^\circ - \theta)$ | (6) $\sin (270^\circ - \theta)$ |

9 If $\cos \theta = -\frac{3}{5}$ where $180^\circ < \theta < 270^\circ$, find the value of each of :

- | | | |
|---------------------------------|--------------------------------|---------------------------------|
| (1) $\csc (180^\circ + \theta)$ | (2) $\sec (-\theta)$ | (3) $\tan (360^\circ - \theta)$ |
| (4) $\cot (\theta - 90^\circ)$ | (5) $\sec (90^\circ + \theta)$ | (6) $\tan (270^\circ - \theta)$ |

10 Find one of the values of θ , where $0^\circ < \theta < 90^\circ$, which satisfies each of the following :

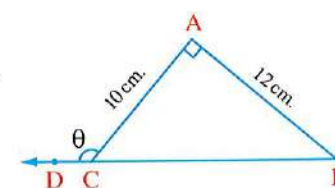
- (1) $\sin (3\theta + 15^\circ) = \cos (2\theta - 5^\circ)$ « 16° »
- (2) $\sec (\theta + 25^\circ) = \csc (\theta + 15^\circ)$ « 25° »
- (3) $\tan (\theta + 20^\circ) = \cot (3\theta + 30^\circ)$ « 10° »
- (4) $\cos \left(\frac{\theta + 20^\circ}{2}\right) = \sin \left(\frac{\theta + 40^\circ}{2}\right)$ « 60° »
- (5) $\tan (\theta + 18^\circ 24') = \cot (\theta + 52^\circ 10')$ « $9^\circ 43'$ »

11 Without using the calculator, choose the correct answer :

(1) In the opposite figure :

$D \in \overline{BC}$, $AC = 10$ cm., $AB = 12$ cm., then $\cot \theta = \dots\dots\dots$

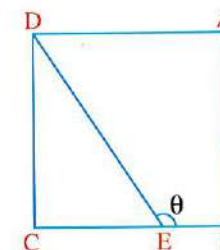
- (a) $\frac{6}{5}$ (b) $-\frac{6}{5}$
(c) $\frac{5}{6}$ (d) $-\frac{5}{6}$



(2) In the opposite figure :

ABCD is a square, $CE = 2 BE$, then $\tan \theta = \dots\dots\dots$

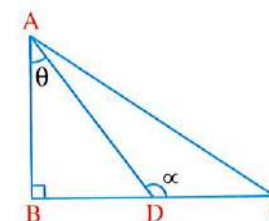
- (a) $-\frac{3}{2}$ (b) $-\frac{2}{3}$
(c) $\frac{1}{2}$ (d) $\frac{2}{3}$



(3) In the opposite figure :

$\triangle ABC$ is a right-angled triangle at B, $\tan \theta = \frac{3}{4}$, then $\cos \alpha = \dots\dots\dots$

- (a) $\frac{3}{4}$ (b) $-\frac{3}{4}$
(c) $-\frac{4}{5}$ (d) $-\frac{3}{5}$

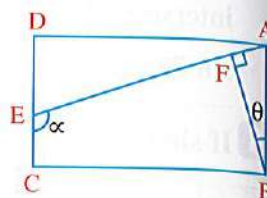


(4) In the opposite figure :

ABCD is a rectangle, $\tan \theta = \frac{1}{3}$, $\overline{BF} \perp \overline{AE}$,

then $\cot \alpha = \dots\dots\dots$

- (a) $\frac{1}{3}$ (b) $\frac{3}{4}$
(c) $-\frac{1}{3}$ (d) $\frac{2}{3}$

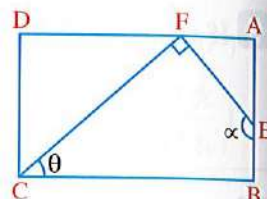


(5) In the opposite figure :

ABCD is a rectangle, $\cos \theta = \frac{3}{4}$, $\overline{EF} \perp \overline{FC}$,

then $\cos \alpha = \dots\dots\dots$

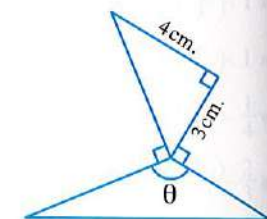
- (a) $\frac{3}{5}$ (b) $-\frac{4}{5}$
(c) $-\frac{3}{4}$ (d) $\frac{3}{4}$



(6) In the opposite figure :

$\cos \theta = \dots\dots\dots$

- (a) $\frac{3}{5}$ (b) $-\frac{3}{5}$
(c) $-\frac{4}{3}$ (d) $-\frac{4}{5}$



(7) In the opposite figure :

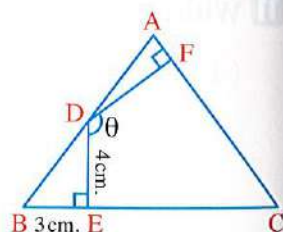
ABC is an isosceles triangle in which

$AB = AC$, $D \in \overline{AB}$, $\overline{DE} \perp \overline{BC}$, $\overline{DF} \perp \overline{AC}$

, $m(\angle EDF) = \theta$, $DE = 4$ cm., $BE = 3$ cm.

, then $\cos \theta = \dots\dots\dots$

- (a) $\frac{3}{5}$ (b) $-\frac{3}{5}$ (c) $-\frac{4}{5}$ (d) $\frac{4}{5}$

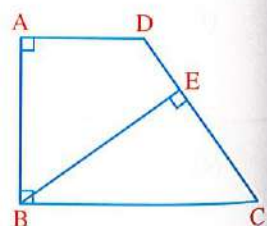


(8) In the opposite figure :

If $3 BE = 4 CE$

, then $\tan(\angle ADC) = \dots\dots\dots$

- (a) $\frac{4}{3}$ (b) $-\frac{4}{3}$
(c) $\frac{3}{4}$ (d) $-\frac{3}{4}$



12 Find the general solution for each of the following equations :

(1) $\sin 2\theta = \cos \theta$

(2) $\cos 5\theta = \sin \theta$

13 Find the values of θ in the following cases where $\theta \in]0, \frac{\pi}{2}]$:

(1) $\csc(\theta + 15^\circ) = \sec 42^\circ$

(2) $\sin(\theta + 30^\circ) = \cos \theta$

(3) $\sin \theta - \cos \theta = 0$

(4) $\csc\left(\theta - \frac{\pi}{6}\right) = \sec \theta$

(5) $\tan(\theta + 27^\circ) = \cot 2\theta$

(6) $\tan(\theta + 10^\circ) = \cot(4\theta - 10^\circ)$

(7) $\sec(2\theta + 35^\circ) = \csc(3\theta - 10^\circ)$

(8) $\sec \theta = \csc(3\theta - 90^\circ)$

(9) $\sin(4\theta + 48^\circ) = \cos(\theta - 33^\circ)$

(10) $\csc 8\theta = \sec 2\theta$

14 Find all values of θ , where $\theta \in]0, \frac{\pi}{2}]$ which satisfies each of the following equations :

(1) $\tan \theta - 1 = 0$

(2) $2 \cos \theta - 1 = 0$

(3) $2 \cos\left(\frac{\pi}{2} - \theta\right) = 1$

(4) $2 \sin\left(\frac{\pi}{2} - \theta\right) = \sqrt{3}$

15 Find the S.S. of each of the following equations knowing that $\theta \in]0, 2\pi[$:

(1) $2 \cos \theta + 1 = 0$

(2) $\sec \theta - \sqrt{2} = 0$

(3) $2 \sin \theta - \sqrt{3} = 0$

(4) $\cos \theta + 1 = 0$

(5) $2 \sin \theta + \sqrt{3} = 0$

(6) $\tan \theta + 1 = 0$

(7) $\sqrt{3} \csc \theta = -2$

(8) $\sin^2 \theta = \frac{1}{4}$

16 If $\cos\left(\frac{3\pi}{2} - \theta\right) = \frac{\sqrt{3}}{2}$, $\sin\left(\frac{\pi}{2} + \theta\right) = \frac{1}{2}$

, find the measure of the smallest positive angle θ

« 300° »

17 If $\sin(2\theta + 15^\circ) = \cos(\theta + 30^\circ)$, where $0^\circ < \theta < 90^\circ$

, find the value of : $\csc^2 2\theta + \cot^2 3\theta + \sec^2 4\theta$

« 9 »

18 If $\frac{\sin(3\theta - 25^\circ)}{\cos(2\theta - 35^\circ)} = 1$, find the value of θ , where $\theta \in]0, \frac{\pi}{4}]$

, then find the value of : $\frac{\sin 18^\circ}{\cos 72^\circ} + \sin(180^\circ - \theta)$

« 30°, 1½ »

19 If $\frac{\tan \theta}{\cot 2\theta} = 1$ where $0^\circ < \theta < 90^\circ$, find the value of θ , then find the value of :

$\sin(180^\circ - 3\theta) \cos(360^\circ - 2\theta) + \tan 2\theta \cot(\theta - 180^\circ)$

« 30°, 3½ »

20 If $\tan(\theta - 15^\circ) = \cot(2\theta + 15^\circ)$ where $\theta \in]0, \frac{\pi}{2}]$

, find the value of θ , then prove that : $\frac{1 + \sin(270^\circ + 2\theta)}{1 + \sin(90^\circ + 2\theta)} = \frac{1}{3}$

« 30° »

21 If $\cos \theta = \frac{3}{5}$ where $270^\circ < \theta < 360^\circ$,

find the value of : $\sin(180^\circ - \theta) + \tan(90^\circ - \theta) - \tan(270^\circ - \theta)$

« -4/5 »

- 22 If $13 \cos \theta = 12$ where $90^\circ < \theta < 360^\circ$,

find the value of : $13 \sin (180^\circ - \theta) - 10 \sin^2 45^\circ \tan^2 60^\circ + 50 \sin 150^\circ$ « 5 »

- 23 If $15 \tan \theta + 8 = 0$, $90^\circ < \theta < 180^\circ$, find the values of the trigonometric functions of the angle θ , then find the value of each of : $2 \sin \theta \cos \theta$, $\sec (1080^\circ + \theta)$ « $-\frac{240}{289}$, $-\frac{17}{15}$ »

- 24 If $\sin \theta = \frac{\sqrt{2}}{2}$, where $\theta \in]0, \frac{\pi}{2}[$, find the value of θ , then :

(1) Find the value of : $\frac{1 - 2 \cot (270^\circ - \theta)}{1 + \cos^2 (270^\circ + \theta)}$

(2) Prove that : $\cos 2\theta = \frac{1 - \tan^2 (270^\circ - \theta)}{\csc^2 (90^\circ + \theta)}$ « $45^\circ, -\frac{2}{3}$ »

- 25 If $B(-5k, -12k)$ is the point of intersection of the terminal side of the directed angle of measure θ in its standard position with the unit circle, $180^\circ < \theta < 270^\circ$, find the value of : $\csc (90^\circ - \theta) \sin (90^\circ + \theta) + 12 \tan (270^\circ + \theta)$ « -4 »

- 26 If $13 \sin \theta - 5 = 0$ where $\theta \in]\frac{\pi}{2}, \pi[$,

find the value of each of : $\csc (270^\circ + \theta)$, $\cos (\theta - 270^\circ)$, $\tan (270^\circ + \theta)$,

then prove that : $\sin (270^\circ - \theta) \times \sec (270^\circ + \theta) \times \cot (270^\circ + \theta) = \sin 90^\circ$

- 27 If $\cos^2 \alpha = \frac{9}{25}$, where $90^\circ < \alpha < 180^\circ$, find the value of : $25 \sin \alpha - 4 \cot \alpha$ « 23 »

- 28 If $\tan \alpha = \frac{3}{4}$ where α is the smallest positive angle, $\tan \beta = \frac{5}{12}$ where $180^\circ < \beta < 270^\circ$, find the trigonometric functions for each of the two angles α, β ,

then find the value of : $\sin \alpha \cos \beta - \cos \alpha \sin \beta$ « $-\frac{16}{65}$ »

- 29 If $\sin \alpha = \frac{3}{5}$ where $\alpha \in]\frac{\pi}{2}, \pi[$, $13 \cos \beta - 5 = 0$ where $\beta \in]\frac{3\pi}{2}, 2\pi[$,

find the value of : $\cos \alpha \cos \beta + \sin \alpha \sin \beta$ « $-\frac{56}{65}$ »

- 30 If $25 \sin \alpha + 24 = 0$ where $180^\circ < \alpha < 270^\circ$, $5 \tan \beta + 12 = 0$

where β is the greatest positive angle, $\beta \in]0^\circ, 360^\circ[$,

find the value of :

(1) $\sin (180^\circ + \alpha) + \cos (180^\circ - \beta)$

(2) $\csc (180^\circ + \alpha) \cot (90^\circ - \beta) - \sec (360^\circ + \alpha) \tan (360^\circ - \beta)$

(3) $\csc (90^\circ + \alpha) \cot (270^\circ + \beta) \tan (270^\circ - \alpha) \csc (270^\circ + \beta)$ « $\frac{187}{325}, \frac{85}{14}, 6\frac{1}{2}$ »

- 31 If the terminal side of the angle whose measure is $(90^\circ - \theta)$ intersects the unit circle at the point $(\frac{5}{13}, y)$, find the trigonometric functions for the angle θ where $\theta \in]0, \frac{\pi}{2}[$

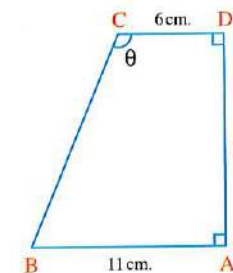
- 32 In the opposite figure :

ABCD is a trapezium, $m(\angle A) = m(\angle D) = 90^\circ$

, $CD = 6$ cm, $AD = 12$ cm, $AB = 11$ cm.

Find : $\sin \theta$

« $\frac{12}{13}$ »

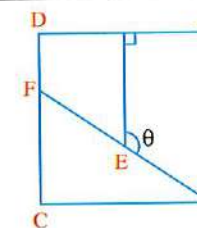


- 33 In the opposite figure :

ABCD is a square, $2 DF = FC$

Find : $\csc \theta$

« $\frac{\sqrt{13}}{3}$ »



Discover the error

- 34 In one of the mathematical competitions, the teacher asked Karim and Ziad to find the value of $\sin(\theta - \frac{\pi}{2})$, then who of them has a correct answer? Explain your answer.

Karim's answer

$$\begin{aligned}\sin\left(\theta - \frac{\pi}{2}\right) &= \sin\left(2\pi + \theta - \frac{\pi}{2}\right) \\ &= \sin\left(\frac{3}{2}\pi + \theta\right) \\ &= -\cos \theta\end{aligned}$$

Ziad's answer

$$\begin{aligned}\sin\left(\theta - \frac{\pi}{2}\right) &= \sin\left[-\left(\frac{\pi}{2} - \theta\right)\right] \\ &= -\sin\left(\frac{\pi}{2} - \theta\right) \\ &= -(-\cos \theta) = \cos \theta\end{aligned}$$



Problems that measure high standard levels of thinking

- 35 Choose the correct answer from those gives :

(1) $\cos 45^\circ \times \cos 46^\circ \times \cos 47^\circ \times \dots \times \cos 135^\circ = \dots$

- (a) zero (b) -1 (c) 1 (d) $\frac{\sqrt{3}}{2}$

(2) $\sin 75^\circ \times \cos 12^\circ \times \sec 15^\circ \times \csc 78^\circ = \dots$

- (a) $1 + \sqrt{2}$ (b) $\sqrt{3} - 1$ (c) 2 (d) 1

- (3) The points A, B, C are placed on the coordinate system where

A(0, 0), B(4, 1), C(0, -2), then $\sin(\angle BAC) = \dots$

- (a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $\frac{4}{\sqrt{17}}$ (d) $-\frac{4}{\sqrt{17}}$

(4) $\frac{\sec 1^\circ \times \sec 2^\circ \times \dots \times \sec 88^\circ \times \sec 89^\circ}{\csc 1^\circ \times \csc 2^\circ \times \dots \times \csc 88^\circ \times \csc 89^\circ} = \dots$

- (a) zero (b) -1 (c) 1 (d) 90

(5) $\frac{\sin(60\pi + \theta) + \cos(90\pi + \theta)}{\cos\left(\frac{5\pi}{2} + \theta\right) - \sin\left(\frac{9\pi}{2} + \theta\right)} = \dots\dots\dots$

- (a) 2 (b) 1 (c) zero (d) -1

(6) If $7X = \frac{\pi}{2}$, then $\frac{\sin 3X}{\cos 4X} + \frac{\tan 2X}{\cot 5X} = \dots\dots\dots$

- (a) -2 (b) -1 (c) 1 (d) 2

(7) If $X + y = 30^\circ$, then :

First: $\tan(X + 2y) \tan(2X + y) = \dots\dots\dots$

- (a) -1 (b) 1 (c) $\sin(X - y)$ (d) $\cos(X - y)$

Second: $\sin(3X + 2y) + \sin(9X + 8y) = \dots\dots\dots$

- (a) zero (b) 1 (c) $\cos X$ (d) $\cos y$

(8) If $f(X) = \sin 2X$, then $f(\theta) + f\left(\theta + \frac{\pi}{2}\right) + f(\theta + \pi) + f\left(\theta + \frac{3\pi}{2}\right) + \dots + f(\theta + 99\pi) + f\left(\theta + \frac{199\pi}{2}\right) = \dots\dots\dots$

- (a) 1 (b) zero (c) 99 (d) 100

(9) If $\cos^2 \theta = 1$, then $\theta = \dots\dots\dots$ where $n \in \mathbb{Z}$

- (a) $n\pi$ (b) $\frac{n}{2}\pi$ (c) $2n\pi$ (d) $(2n + 1)\pi$

(10) The number of solutions of the equation : $\tan X = -\sqrt{3}$ where $0 \leq X \leq 15\pi$ is $\dots\dots\dots$

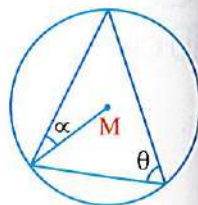
- (a) 2 (b) 4 (c) 15 (d) 30

(11) In the opposite figure :

M is the centre of the circle

, then $\tan \theta = \dots\dots\dots$

- (a) $\tan \alpha$ (b) $\cot \alpha$ (c) $\cos \alpha$ (d) $\sin \alpha$

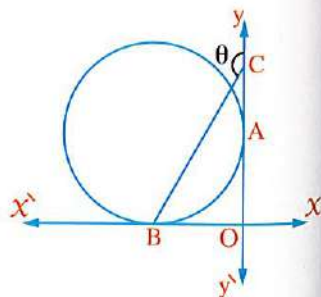


(12) In the opposite figure :

If A(0, 3), C(0, 4)

, then $\cos \theta = \dots\dots\dots$

- (a) $-\frac{4}{5}$ (b) $\frac{3}{4}$
(c) $-\frac{3}{5}$ (d) $-\frac{3}{4}$

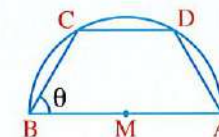


(13) In the opposite figure :

\overline{AB} is a diameter of the semi-circle M

and $13 \sin \theta = 12$, then $\cos(\angle ADC) = \dots\dots\dots$

- (a) $-\frac{12}{13}$ (b) $-\frac{5}{13}$ (c) $\frac{5}{13}$ (d) $\frac{12}{13}$



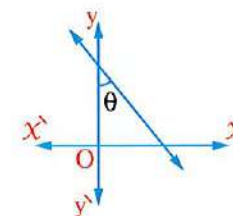
(14) In the opposite figure :

If the equation of the straight line is $y = -\frac{3}{4}x + 5$

, θ is an acute angle between

the straight line and y-axis, then $\dots\dots\dots$

- (a) $\cos \theta = \frac{3}{4}$ (b) $\sin \theta = \frac{4}{3}$ (c) $\tan \theta = \frac{4}{3}$ (d) $\sin \theta = \frac{3}{5}$



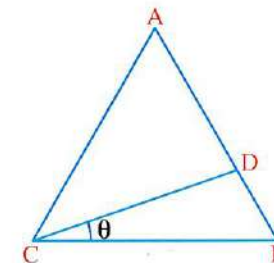
(15) In the opposite figure :

ABC is an equilateral triangle

, $D \in \overline{AB}$ such that : $2AD = 3BD$

, then $\tan \theta = \dots\dots\dots$

- (a) $\frac{2}{3}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{\sqrt{3}}{5}$ (d) $\frac{2}{5}$



36 ABC is an obtuse-angled triangle at C, $\sin C = \frac{3}{5}$

Find : $\sin(A + B + 2C)$

« $-\frac{3}{5}$ »

37 Find the value of each of :

(1) $\cos 20^\circ + \cos 40^\circ + \cos 60^\circ + \dots + \cos 160^\circ + \cos 180^\circ$

« -1 »

(2) $\sin 1^\circ + \sin 2^\circ + \sin 3^\circ + \dots + \sin 358^\circ + \sin 359^\circ$

« zero »

11

Graphing trigonometric functions



Test

yourself

From the school book

1 Choose the correct answer :

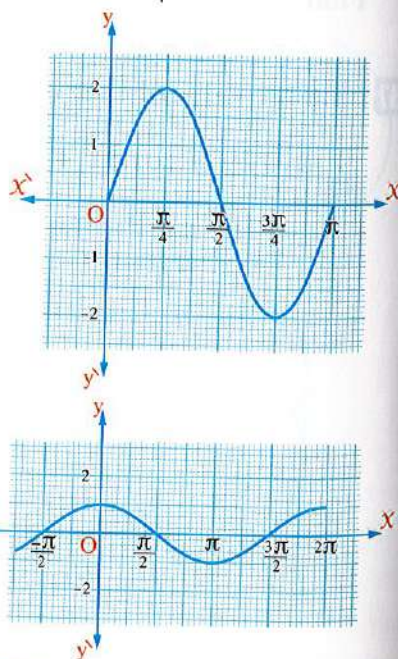
- (1) The range of the function $f : f(\theta) = \sin \theta$ is
 (a) $\{-1, 1\}$ (b) $[-1, 1]$ (c) $]-1, 1[$ (d) $]-\infty, \infty[$
- (2) If $f(\theta) = \cos 5\theta$, then the range of the function is
 (a) $\{-5, 5\}$ (b) $[-1, 1]$ (c) $]-5, 5[$ (d) $[-5, 5]$
- (3) The maximum value of the function $g : g(\theta) = 4 \sin \theta$ is
 (a) 4 (b) 1 (c) zero (d) ∞
- (4) The minimum value of the function $h : h(\theta) = 5 \cos 7\theta$ is
 (a) 5 (b) zero (c) -5 (d) -7
- (5) The function $f : f(\theta) = 2 \sin 4\theta$ is a periodic function and its period equals
 (a) 2π (b) π (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

- (6) The opposite figure represents one cycle of the trigonometric function $y = f(x)$, then the rule of the function is

- (a) $y = 2 \sin x$ (b) $y = \sin 2x$
 (c) $y = 2 \sin 2x$ (d) $y = \sin x$

- (7) The opposite figure represents the curve of the trigonometric function $y = f(x)$ then the rule of the function is

- (a) $y = \sin \theta$ (b) $y = \cos \theta$
 (c) $y = 2 \cos \theta$ (d) $y = 2 \sin \theta$



2 Find the maximum and minimum values, then write the range of each of the following functions :

(1) $y = \frac{1}{2} \sin \theta$

(2) $y = \frac{1}{3} \sin 2\theta$

(3) $y = 2 \sin 3\theta$

3 Represent graphically each of the following functions and from the graph determine the minimum and maximum values of the function and write the range :

(1) $y = 4 \cos \theta$

where $\theta \in [0, 2\pi]$

(2) $y = 4 \sin \theta$

where $\theta \in [0, 2\pi]$

(3) $y = 2 \cos \theta$

where $\theta \in [-2\pi, 2\pi]$

(4) $y = 3 \sin \theta$

where $\theta \in [-2\pi, 2\pi]$

4 Represent graphically each of the following functions, and from the graph determine the minimum and maximum values of the function, and write the range :

(1) $y = \cos 3\theta$

where $0^\circ \leq \theta \leq 120^\circ$

(2) $y = 5 \sin 2\theta$

where $0^\circ \leq \theta \leq 180^\circ$

5 Use the graph calculator or graphing program on your computer to graph each of the functions : $y = 4 \cos \theta$, $y = 3 \sin \theta$, then find from the graph :

- (1) The range of the function.
 (2) The maximum and minimum values of the function.



Problems that measure high standard levels of thinking

6 Choose the correct answer from those givens :

(1) If $\frac{2 - \sin x}{3} = m$, then

(a) $\frac{1}{3} \leq m \leq 1$

(b) $\frac{2}{3} \leq m \leq 3$

(c) $1 \leq m \leq 3$

(d) $2 \leq m \leq 4$

(2) The function $y = \sin\left(\frac{\pi}{4} + x\right)$ has maximum value at $x = \dots\dots\dots$

(a) $\frac{\pi}{2}$

(b) $-\frac{\pi}{2}$

(c) $\frac{\pi}{4}$

(d) zero

(3) The function $f : f(x) = \sin(bx)$ is a periodic function its period $\frac{2\pi}{3}$, then $b = \dots\dots\dots$

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) 3

(d) 6

(4) If the two points $(x_1, \cos x_1)$, $(x_2, \cos x_2)$ lie on the curve of the function $f(x) = \cos x$, then the greatest value of the expression $(\cos x_1 - \cos x_2) = \dots\dots\dots$

(a) 1

(b) 2

(c) zero

(d) 180°

(5) If the function $f : f(x) = a \cos bx$ where $a > 0$ is a periodic function and its period $\frac{\pi}{2}$ and its range $[-1, 1]$, then $\frac{a}{b} = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) $\frac{-1}{4}$ (c) $\frac{-1}{2}$ (d) $\frac{1}{4}$

(6) The opposite figure represents the curve $y = \sin x$, then $|a| + |b| = \dots\dots\dots$

- (a) 1 (b) 2
(c) π (d) 2π

(7) The opposite figure represents the curve $y = 3 \sin \frac{1}{2}x$, then the x -coordinate of B equals $\dots\dots\dots$

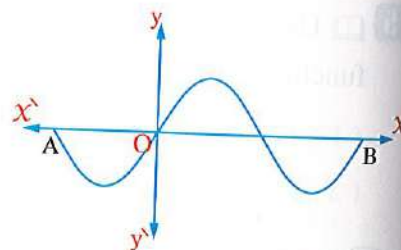
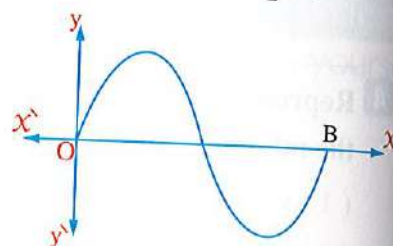
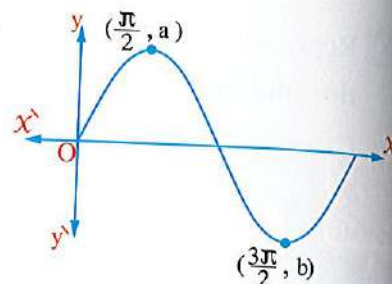
- (a) $\frac{\pi}{2}$ (b) π
(c) 2π (d) 4π

(8) In the opposite figure: If $y = \sin x$, then $B - A = \dots\dots\dots$

- (a) π (b) 2π
(c) 3π (d) 4π

(9) The number of intersections of the curve $y = \sin 3x$ with x -axis in the interval $[0, 2\pi]$ equals $\dots\dots\dots$

- (a) 2 (b) 3 (c) 4 (d) 7



Exercise

12

Finding the measure of an angle given the value of one of its trigonometric ratios ?

From the school book



Test yourself

1 Choose the correct answer from those given :

- (1) If $\theta = \sin^{-1} \frac{\sqrt{3}}{2}$, then $\theta = \dots\dots\dots$
 (a) 60° (b) 120° (c) 240° (d) 300°
- (2) If $\tan \theta = \frac{-1}{\sqrt{3}}$, $90^\circ < \theta < 180^\circ$, then $\theta = \dots\dots\dots$
 (a) 30° (b) 120° (c) 150° (d) 210°
- (3) If $\csc \theta = -2$, $270^\circ < \theta < 360^\circ$, then $\theta = \dots\dots\dots$
 (a) 30° (b) 300° (c) 330° (d) 150°
- (4) $\sin^{-1} 0.7 \approx \dots\dots\dots$
 (a) $44^\circ 25' 37''$ (b) $135^\circ 34' 23''$ (c) $224^\circ 25' 37''$ (d) $315^\circ 34' 23''$
- (5) $\sin^{-1} (-0.6) \approx \dots\dots\dots$
 (a) -36.87° (b) 143.13° (c) 216.87° (d) 323.13°
- (6) If $\cos \theta = 0.436$, where θ is the measure of the smallest positive angle, then $\theta = \dots\dots\dots$
 (a) $64^\circ 9'$ (b) $115^\circ 51'$ (c) $244^\circ 9'$ (d) $295^\circ 51'$
- (7) If $\tan \theta = 1.8$ and $90^\circ \leq \theta \leq 360^\circ$, then $\theta = \dots\dots\dots$
 (a) $60^\circ 57'$ (b) $119^\circ 3'$ (c) $240^\circ 57'$ (d) $299^\circ 3'$

2 Find in degrees the measure of the smallest positive angle θ satisfying :

- (1) $\sin \theta = 0.6$ (2) $\cos \theta = 0.7865$ (3) $\tan \theta = 2.4577$
 (4) $\tan \theta = -0.8227$ (5) $\sin \theta = -0.4652$ (6) $\cos \theta = -0.5206$

(7) $\cot \theta = 3.6218$	(8) $\cot \theta = -1.4612$	(9) $\sec \theta = 1.0478$
(10) $\csc \theta = -2.5466$	(11) $\sec \theta = -3.57$	(12) $\csc \theta = 2.9811$

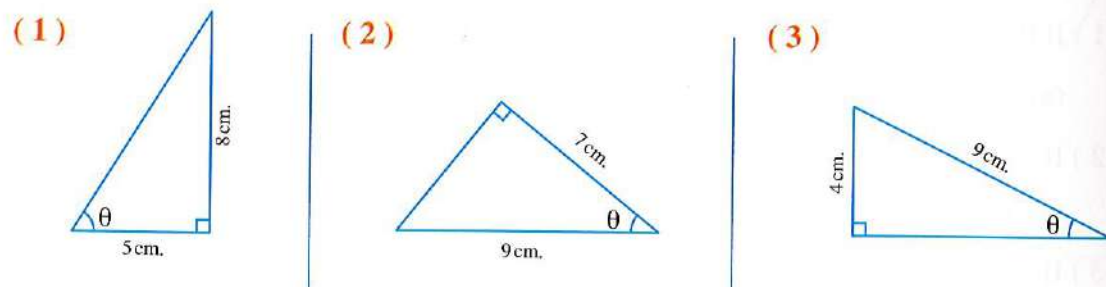
3 If $0^\circ < \theta < 360^\circ$, find θ which satisfies each of the following :

(1) $\sin \theta = 0.86603$	(2) $\cos \theta = -0.4752$	(3) $\csc \theta = -1.2576$
(4) $\tan \theta = 1.5417$	(5) $\cos \theta = -0.642$	(6) $\sec \theta = 2.0515$
(7) $\csc \theta = -1.8715$	(8) $\cot \theta = -2.7012$	(9) $\tan \theta = -2.1456$

4 If the terminal side of angle θ in the standard position intersects the unit circle at point B, then find $m(\angle \theta)$ where $0^\circ < \theta < 360^\circ$ when :

(1) $B\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	(2) $B\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	(3) $B\left(\frac{6}{10}, -\frac{8}{10}\right)$
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5 Find the degree measure of the angle θ in each of the following figures :



6 If $\sin \theta = \frac{1}{3}$ and $90^\circ \leq \theta \leq 180^\circ$:

- (1) Calculate the measure of the angle θ to the nearest second.
 (2) Find the value of each of the following : $\cos \theta$, $\tan \theta$, $\sec \theta$

7 ABC is a triangle in which $\cos A = -0.5807$, $\tan B = 0.4578$

Find to the nearest minute $m(\angle C)$

« $29^\circ 54'$ »

8 If $0^\circ < \theta < 360^\circ$, find the values of θ in degrees and minutes which satisfy :

$\tan \theta = \sin 23^\circ 48' + \cos 84^\circ 32'$ « $26^\circ 31'$ or $206^\circ 31'$ »

9 If $0^\circ < \theta < 360^\circ$, find the values of θ in degrees and minutes which satisfy :

$\cos \theta = \sin 70^\circ - 2 \cos 80^\circ \tan 75^\circ$ « $110^\circ 53'$ or $249^\circ 7'$ »

10 If $\tan \theta = \frac{4}{3}$ where θ is the measure of the greatest positive angle $\theta \in]0, 2\pi[$

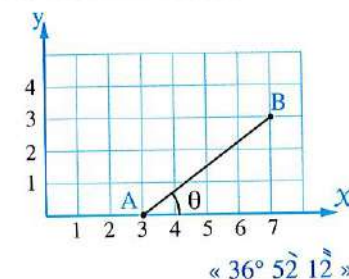
Find the value of α to the nearest minute if :

$\sin \alpha = \sin 150^\circ \sin(-\theta) + \frac{1}{5} \csc(180^\circ + \theta) \tan 225^\circ$ « $40^\circ 32'$ or $139^\circ 28'$ »

11 If $\sin \alpha = \frac{3}{5}$ where $90^\circ < \alpha < 180^\circ$, find θ from the equation :
 $-\frac{5}{4} \cos(360^\circ - \alpha) + \cot(270^\circ - \theta) = 2$ where $0^\circ < \theta < 360^\circ$

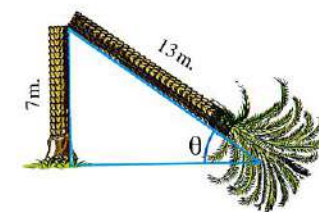
« 45° or 225° »

12 The opposite figure represents a line segment joining between the two points A(3, 0), B(7, 3). Find the measure of the angle θ included between \overline{AB} and the X-axis.



Discover the error

13 A palm of length 20 metres was broken due to the wind as in the opposite figure, if the length of the vertical part equals 7 metres, and the inclined part is of length 13 metres and θ is the angle which the inclined part makes with the horizontal, find in degrees the measure of θ



Karim's answer

$$\begin{aligned} \therefore \csc \theta &= \frac{13}{7} \\ \therefore \theta &= \csc^{-1} \frac{13}{7} \\ \therefore m(\angle \theta) &= 32^\circ 34' 44'' \end{aligned}$$

Omar's answer

$$\begin{aligned} \therefore \sec \theta &= \frac{13}{7} \\ \therefore \theta &= \sec^{-1} \frac{13}{7} \\ \therefore m(\angle \theta) &= 57^\circ 25' 16'' \end{aligned}$$

Which answer is right? Why?



Problems that measure high standard levels of thinking

14 Choose the correct answer from those givens :

(1) In the opposite figure :

$m(\angle ABC) = \dots\dots\dots$

(a) $\sin^{-1}\left(\frac{3}{4}\right)$

(b) $\sin^{-1}\left(\frac{4}{3}\right)$

(c) $\tan^{-1}\left(\frac{3}{4}\right)$

(d) $\cot^{-1}\left(\frac{3}{4}\right)$

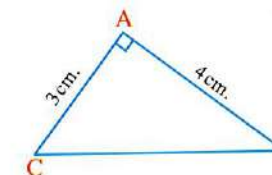
(2) $\sin\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \dots\dots\dots$

(a) $\frac{\sqrt{3}}{2}$

(b) $\frac{1}{2}$

(c) 30°

(d) 60°



(3) $\csc(\cos^{-1}(\text{zero})) = \dots\dots\dots$

(a) 1

(b) -1

(c) $\frac{\pi}{2}$

(d) zero

(4) In the opposite figure :

$\sin\left(\tan^{-1}\left(\frac{5}{12}\right)\right) = \dots\dots\dots$

(a) $\frac{5}{12}$

(b) $\frac{5}{13}$

(c) $\frac{12}{13}$

(d) 13

(5) In the opposite figure :

ABCD is a parallelogram , its area = 40 cm^2 , then $m(\angle A) \approx \dots\dots\dots$

(a) 37°

(b) 56°

(c) 53°

(d) 34°

(6) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \cot^{-1}(\sqrt{3}) = \dots\dots\dots$

(a) $\frac{\pi}{3}$

(b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$

(d) $\frac{\pi}{6}$

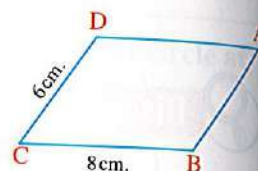
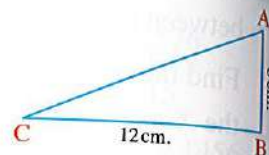
(7) $\cos^{-1} x + \sin^{-1} x \approx \dots\dots\dots$

(a) zero

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) π



Life Applications on Unit Two

From the school book



- One of the gymnasts spins on the play device by an angle of measure 200° . Draw this angle in the standard position , then find its measure in radian. « 3.49^{rad} »
- What is the distance covered by a point on the end of the minute hand in 10 minutes, if the hand length is 6 cm. ? « $2\pi \text{ cm.}$ »
- A satellite revolves around the Earth in a circular path way a full revolution every 6 hours , if the radius length of its path from the center of the Earth is 9000 km. Find its speed in kilometre per hour. « 9424.78 km/hr »
- A satellite spins around the Earth in a circular path a complete revolution every 3 hours. If the radius length of the Earth approximately equals 6400 km. and the distance between the satellite and the surface of the Earth equals 3600 km. , find the distance which the satellite covers during one hour approximating the result to the nearest km.

« 20944 km »

- A sundial is used to determine the time during the day through the shadow length falling on a graduated surface to show the clock and its parts. If the shadow rotates on the disk by the rate 15° every hour.

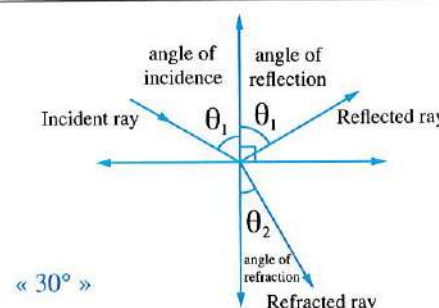


- Find the radian measure of the angle which the shadow rotates from it after 4 hours.
- After how many hours does the shadow rotate by an angle of radian measure $\frac{2\pi}{3}$?
- The radius of a sundial is 24 cm. In terms of π , find the arc length which the rotation of the shadow makes on the edge of the disk after 10 hours.

« 1.05^{rad} , 8 hours , $20\pi \text{ cm.}$ »

- When the sun rays fall on a translucent surface , they are reflected with the same angle of incidence but some rays are refracted when they pass through this surface as shown in the opposite figure.

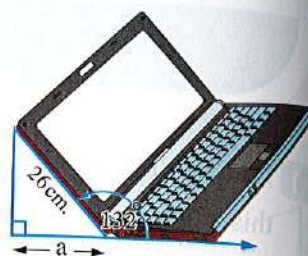
If $\sin \theta_1 = k \sin \theta_2$ and $k = \sqrt{3}$, $\theta_1 = 60^\circ$,
find the measure of angle θ_2

« 30° »

- 7** When Karim uses his laptop, the measure of the angle of inclination of his laptop on the horizontal is 132° as shown in the opposite figure.

(1) Draw the figure on the coordinate plane such that the angle of measure 132° is in the standard position, then find its related angle.

(2) Write a trigonometric function you can use to find the value of a , then find the value of a to the nearest centimetre.



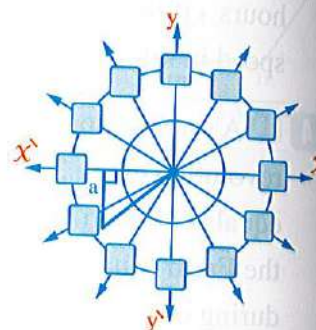
« 17 cm. »

- 8** The spinning wheel is commonly spreading out in the amusement parks. It contains a number of boxes rotating in a circular arc of radius length 12 m.

If the measure of the common angle with the terminal side in the standard position is $\frac{5\pi}{4}$

(1) Draw the angle of measure $\frac{5\pi}{4}$ in the standard position.

(2) Write a trigonometric function you can use to find the value of a , then find the value of a in metre to the nearest hundredth.



« 8.49 m. »

- 9** It is possible for the ships entering the port, if the level of water is high as a result of the movement of the ebb and tide, where the depth of water is at least 10 metres.

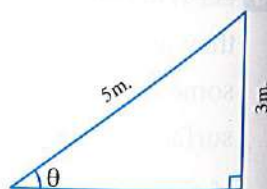
The movement of the ebb and tide in that day is given by the relation,

$S = 6 \sin(15n)^\circ + 10$ where n is the time elapsed after the mid-night in hour according to 24 hours system.

- (1) How many times did the depth of water completely reach 10 metres in the port?
- (2) Draw a graph representation to show how the depth of water vary with the movement of the ebb and tide during the day.
- (3) How many hours during the day at which the ship be able to enter the port?

- 10** A ladder of length 5 metres rests on a wall.

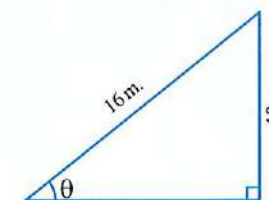
If the height of the ladder from the ground is 3 metres, find in radian the measure of the angle of inclination of the ladder to the horizontal.



« 0.644^{rad} »

- 11** There is a skiing game in the theme parks.

If the height of one of these games is 10 metres, and its length is 16 metres as in the opposite figure, write a trigonometric function you can use to find the value of the angle θ , then find the value of the angle in degrees to the nearest thousands.



« 38.682° »

- 12** Karim descends by his car down a ramp of length 65 m. and its height is 8 m. If the ramp makes an angle θ with the horizontal, find $m(\angle \theta)$ in degree measure.



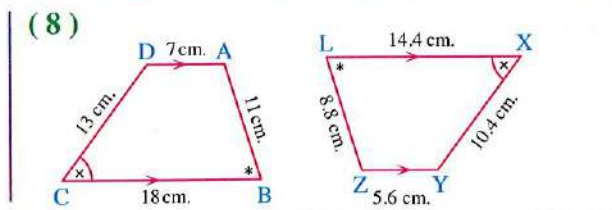
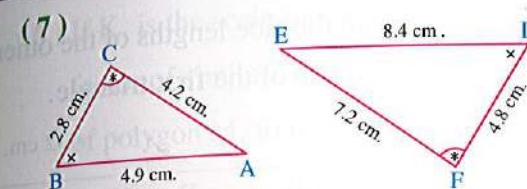
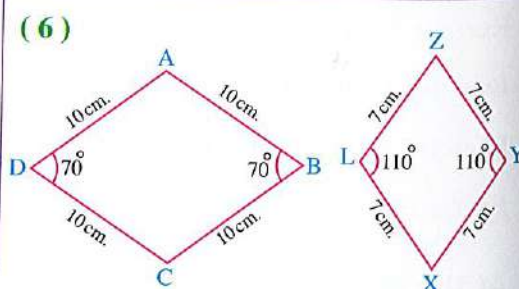
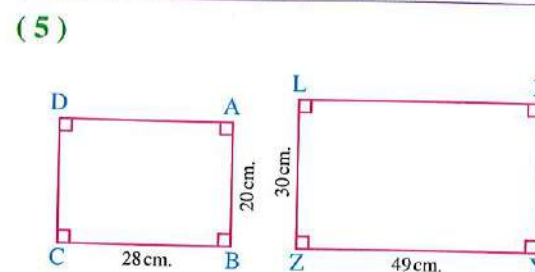
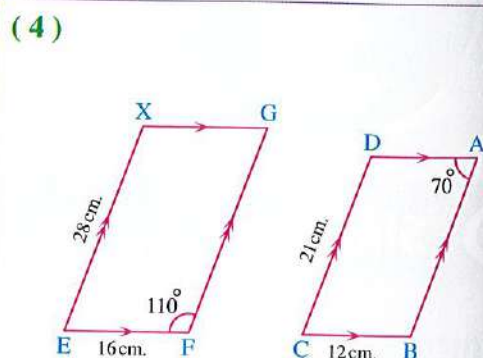
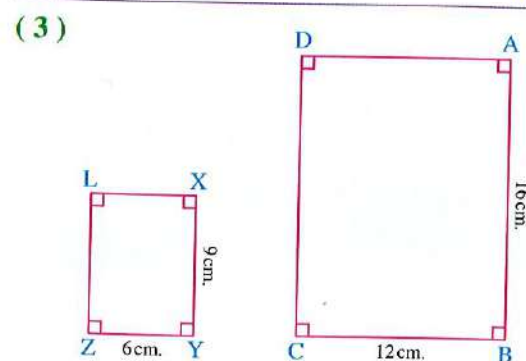
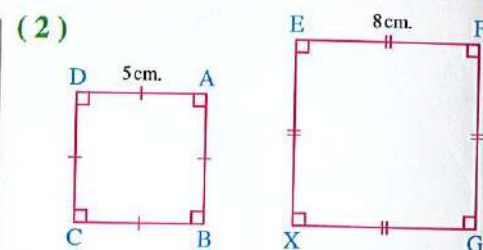
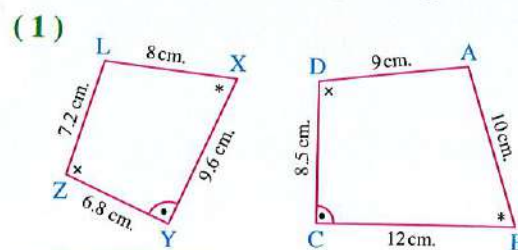
« $7^\circ 4' 11''$ »

Similarity of polygons

From the school book

Test
yourself

- 1 Show which of the following pairs of polygons are similar. Write the similar polygons in the order of their corresponding vertices and determine the similarity ratio :



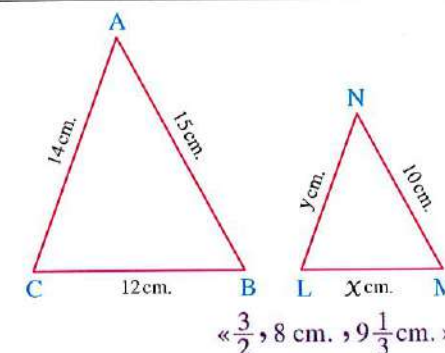
- 2 In the opposite figure :

$\triangle ABC \sim \triangle NML$

The lengths of sides are shown on the figures.

Find :

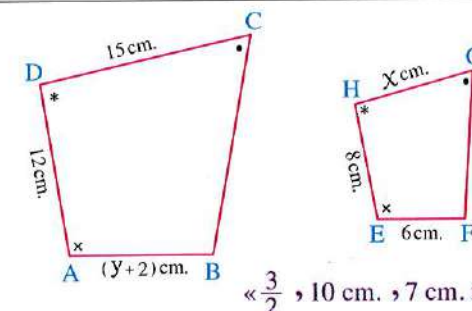
- (1) The scale factor of similarity of triangle ABC to triangle NML
(2) The values of x and y



- 3 In the opposite figure :

Polygon ABCD \sim polygon EFGH

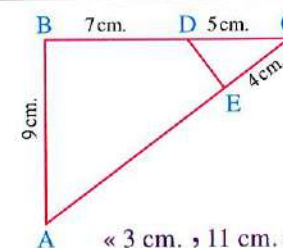
- (1) Find : The scale factor of similarity of polygon ABCD to polygon EFGH
(2) Find the values of : x and y



- 4 In the opposite figure :

$\triangle CBA \sim \triangle CED$

Using the lengths shown on the figure ,
find the length of each of : \overline{ED} and \overline{EA}

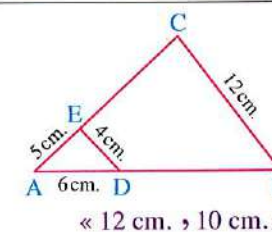


- 5 In the opposite figure :

$\triangle ADE \sim \triangle ABC$

Prove that : $\overline{DE} \parallel \overline{BC}$,

and from the lengths shown on the figure ,
find the length of each of : \overline{BD} and \overline{CE}

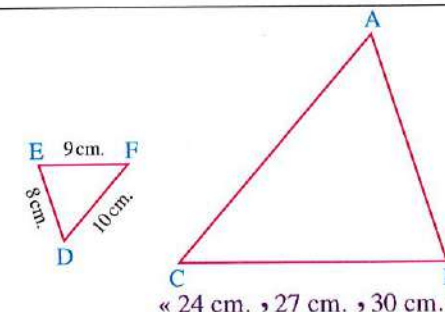


- 6 In the opposite figure :

$\triangle ABC \sim \triangle DEF$

, $DE = 8$ cm. , $EF = 9$ cm. , $FD = 10$ cm.

If the perimeter of $\triangle ABC = 81$ cm.
, find the side lengths of : $\triangle ABC$



- 7 Two similar triangles, the perimeter of the first is 74 cm. and the side lengths of the other are 4.5 cm., 6 cm., and 8 cm. Find the length of the longest side of the first triangle.

« 32 cm. »

- 8 Two similar rectangles, the dimensions of the first are 8 cm. and 12 cm., and the perimeter of the second is 200 cm. Find the length of the second rectangle and its area.

« 60 cm., 2400 cm². »

- 9 In the opposite figure :

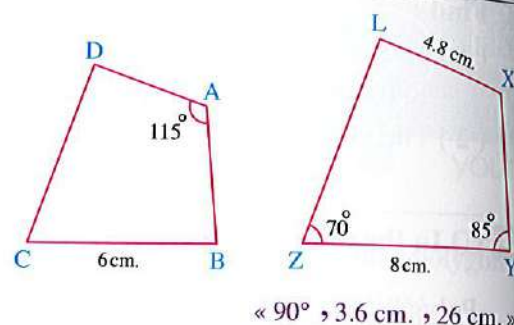
Polygon ABCD ~ polygon XYZL

- (1) Calculate : $m(\angle XLZ)$, length of \overline{AD}

- (2) If the perimeter of the polygon

ABCD = 19.5 cm.

Find : The perimeter of the polygon XYZL



« 90°, 3.6 cm., 26 cm. »

- 10 Choose the correct answer from those given :

- (1) If k is the scale factor of similarity of polygon M_1 to polygon M_2 and $0 < k < 1$, then the polygon M_1 is to polygon M_2

(a) congruent to (b) enlargement (c) minimization (d) of double area

- (2) Two similar polygons, the ratio between the lengths of two corresponding sides in them is 3 : 4, if the perimeter of the smaller is 15 cm., then the perimeter of the bigger is cm.

(a) 20 (b) $\frac{80}{3}$ (c) 27 (d) $\frac{95}{4}$

- (3) To make two polygons M_1 and M_2 similar it is sufficient to get

(a) their corresponding angles are equal in measures only.

(b) their corresponding sides are in proportion only.

(c) (a) and (b) together.

(d) nothing of the previous.

- (4) To make two rhombuses ABCD, XYZL similar it is sufficient to get

(a) $m(\angle A) = 60^\circ$, $m(\angle Y) = 120^\circ$ only.

(b) the perimeter of rhombus ABCD = 2 the perimeter of the rhombus XYZL only.

(c) (a) and (b) together.

(d) nothing of the previous.

- (5) If K_1 is the scale factor of similarity of polygon M_1 to polygon M_2 and K_2 is the scale factor of similarity of polygon M_2 to polygon M_3 , then the scale factor of similarity of polygon M_1 to polygon M_3 is

(a) $K_1 + K_2$ (b) $K_1 K_2$ (c) $\frac{K_1}{K_2}$

(d) $\frac{K_2}{K_1}$

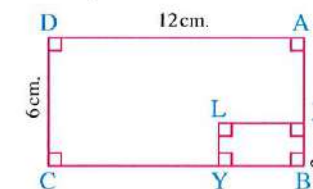
- (6) In the opposite figure :

Rectangle ABCD ~ rectangle XBYL,

then the length of \overline{YC} = cm.

(a) 6 (b) 8 (c) 10

(d) 11



- (7) In the opposite figure :

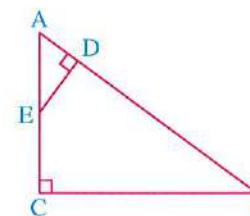
If $\triangle ABC \sim \triangle AED$,

$m(\angle B) = 3x + 10^\circ$, $m(\angle AED) = x + 30^\circ$,

then $m(\angle A)$ =

(a) 50° (b) 40° (c) 30°

(d) 60°



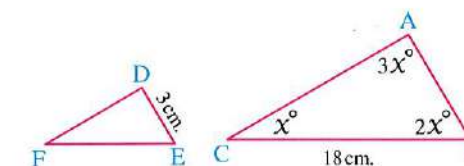
- (8) In the opposite figure :

If $\triangle ABC \sim \triangle DEF$,

then the length of \overline{FE} = cm.

(a) 3 (b) 4 (c) 6

(d) 8



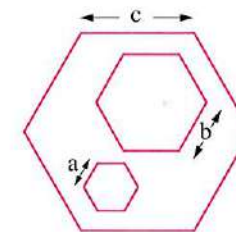
- (9) The opposite figure shows three regular hexagons, the ratio between their sides lengths is as follows

$a : b = 1 : 2$, $b : c = 3 : 8$

If the length of the side of the greatest hexagon = 32 cm.

, then the perimeter of the smallest hexagon = cm.

(a) 12 (b) 6 (c) 36 (d) 48



- 11 If polygon ABCD ~ polygon XYZL, complete :

(1) $\frac{AB}{BC} = \frac{\dots}{YZ}$

(2) $AB \times ZL = XY \times \dots$

(3) $\frac{BC + YZ}{YZ} = \frac{\dots + LX}{LX}$

(4) $\frac{\text{perimeter of polygon } \dots}{\text{perimeter of polygon } \dots} = \frac{XY}{AB}$

- 12 Polygon $ABCD \sim$ polygon $XYZL$, if $AB = 32$ cm., $BC = 40$ cm., $XY = 3m - 1$, $YZ = 3m + 1$, find the numerical value of m

« 3 »

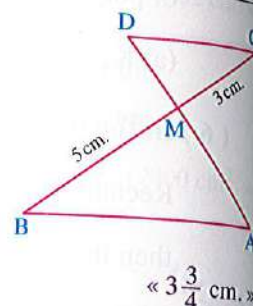
- 13 In the opposite figure :

$$\triangle MAB \sim \triangle MDC$$

Prove that : $\overline{AB} \parallel \overline{CD}$

and if $MC = 3$ cm., $MB = 5$ cm., $AD = 6$ cm.

Find : The length of \overline{AM}

« $3\frac{3}{4}$ cm. »

- 14 In the opposite figure :

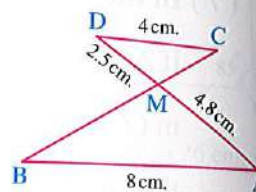
$$\triangle MAB \sim \triangle MCD$$

Prove that : The figure $ABDC$ is a cyclic quadrilateral.

And if $AB = 8$ cm., $CD = 4$ cm., $MA = 4.8$ cm.

, $MD = 2.5$ cm.

Find : The length of \overline{BC}



« 7.4 cm. »

- 15 Triangle ABC has : $AB = 5$ cm., $BC = 6$ cm., $AC = 9$ cm. Find the lengths of the sides of a similar triangle if :

(1) The scale factor of similarity = 2.5

(2) The scale factor of similarity = 0.6

- 16 The dimensions of a rectangle are 10 cm. and 6 cm. Find the perimeter and the area of another rectangle similar to it if :

(1) The scale factor equals 3

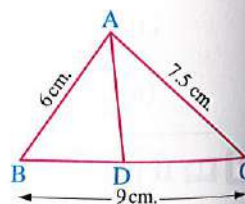
(2) The scale factor equals 0.4

- 17 In the opposite figure :

$$\triangle ABC \sim \triangle DBA$$

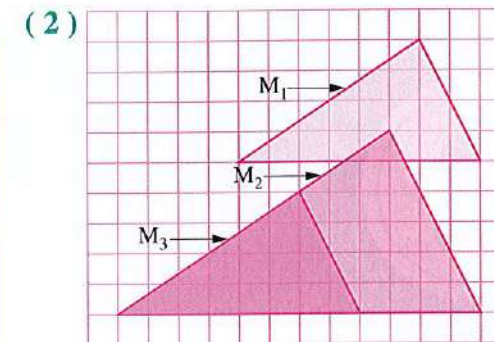
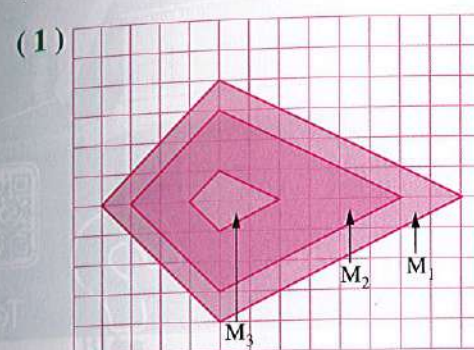
Prove that : \overline{AB} is a tangent to the circle passing through the vertices of $\triangle ADC$ and that AB is a mean proportional between BD and BC and if $AB = 6$ cm., $AC = 7.5$ cm.

Find : The length of each of \overline{AD} , \overline{CD}



« 5 cm., 5 cm. »

- 18 In each of the following figures : Polygon $M_1 \sim$ polygon $M_2 \sim$ polygon M_3
Find the scale factor of similarity of each of polygon M_1 and polygon M_2 with respect to polygon M_3



Problems that measure high standard levels of thinking

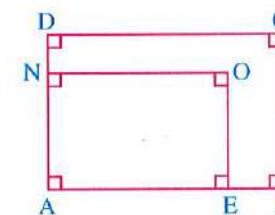
- 19 In the opposite figure :

Rectangle $ABCD \sim$ rectangle $AEON$

Prove that :

Perimeter of rectangle $ABCD$:

perimeter of rectangle $AEON = (AB - AD) : (AE - AN)$



Similarity of triangles

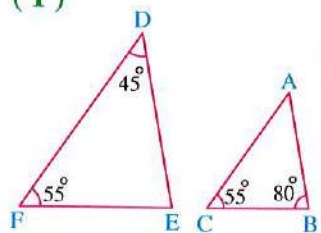


Test

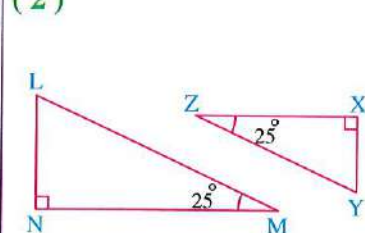
yourself

- 1 State in which of the following cases, the two triangles are similar. In case of similarity, state why they are similar :

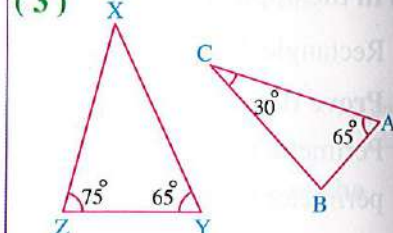
(1)



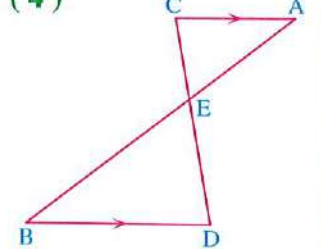
(2)



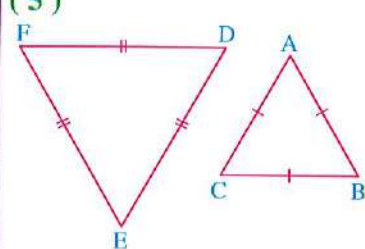
(3)



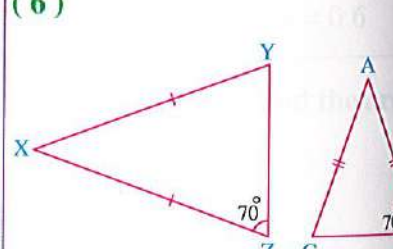
(4)



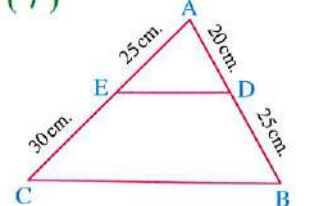
(5)



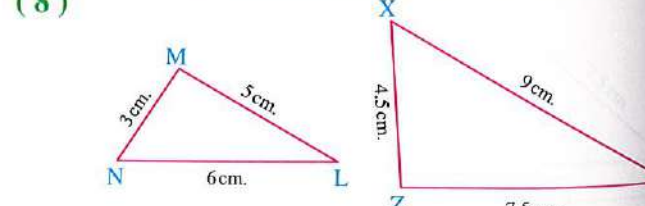
(6)



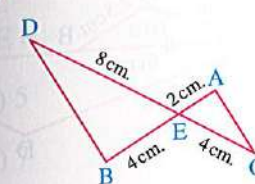
(7)



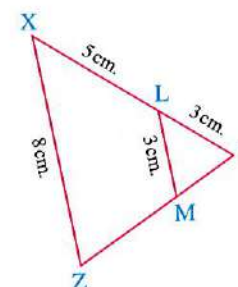
(8)



(9)



(10)



- 2 Choose the correct answer from those given :

- (1) In the opposite figure :

BD = cm.

(a) 5

(b) 6

(c) 4

(d) 7

- (2) In the opposite figure :

If \overline{AB} is a tangent to the circle, then $AB = \dots\dots\dots$ cm.

(a) 4

(b) 5

(c) 6

(d) 7

- (3) In the opposite figure :

$\frac{y}{x} = \dots\dots\dots$

(a) 1

(b) $\frac{4}{3}$ (c) $\frac{3}{4}$

(d) 2

- (4) In the opposite figure :

If $m(\angle ADC) = m(\angle ACB)$

, then $AB = \dots\dots\dots$ cm.

(a) 12

(b) 16

(c) 18

(d) 20

- (5) In the opposite figure :

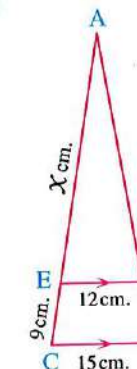
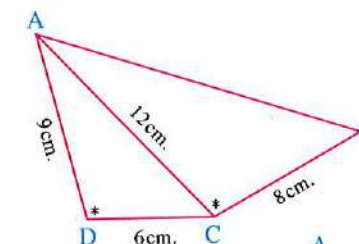
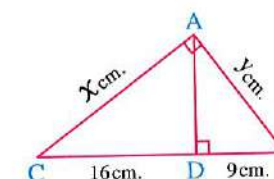
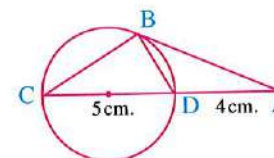
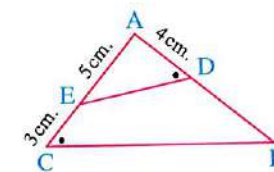
$x = \dots\dots\dots$ cm.

(a) 12

(b) 24

(c) 36

(d) 48

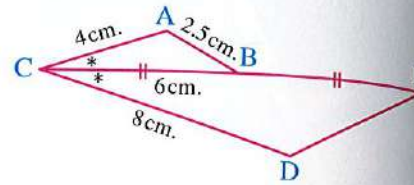


(6) In the opposite figure :

If B is the midpoint of \overline{CE}

, then $DE = \dots\dots\dots$ cm.

- (a) 4 (b) 5
(c) 6 (d) 7

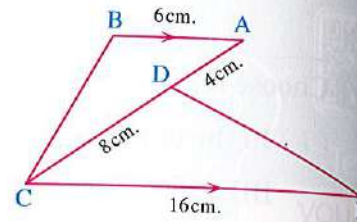


(7) In the opposite figure :

If $\overline{AB} \parallel \overline{EC}$

, then $\frac{ED}{BC} = \dots\dots\dots$

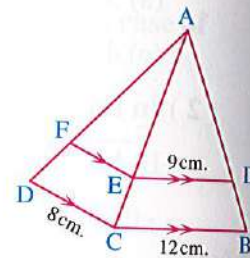
- (a) $\frac{4}{3}$ (b) $\frac{3}{4}$
(c) $\frac{2}{3}$ (d) $\frac{1}{2}$



(8) In the opposite figure :

$EF = \dots\dots\dots$ cm.

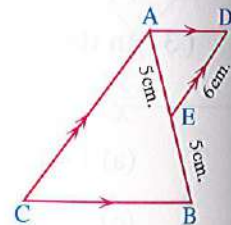
- (a) 3 (b) 6
(c) 9 (d) 12



(9) In the opposite figure :

$AC = \dots\dots\dots$ cm.

- (a) 6 (b) 9
(c) 12 (d) 15

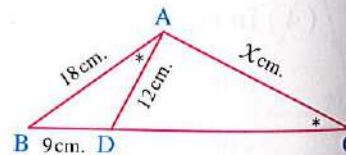


(10) In the opposite figure :

If $m(\angle DAB) = m(\angle C)$

, then $x = \dots\dots\dots$

- (a) 6 (b) 18
(c) 21 (d) 24



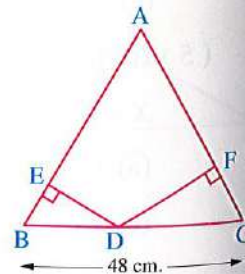
(11) In the opposite figure :

ABC is an isosceles triangle

where $AB = AC$, $BC = 48$ cm.

, $\frac{DE}{DF} = \frac{5}{7}$, then $DC = \dots\dots\dots$ cm.

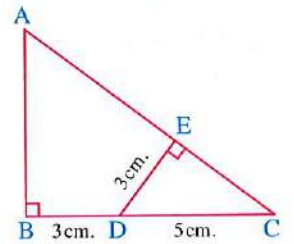
- (a) 12 (b) 20
(c) 24 (d) 28



(12) In the opposite figure :

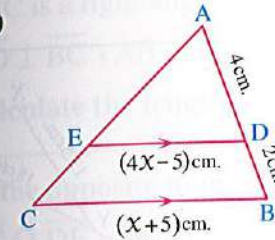
$AE = \dots\dots\dots$ cm.

- (a) 5 (b) 6
(c) 7 (d) 8

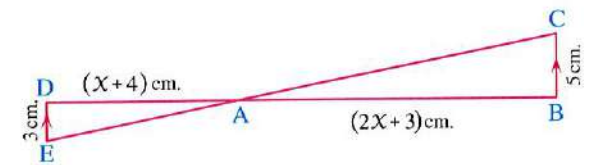


3 In each of the following figures, find the numerical value of the used symbol / symbols in measure. Explain your answer :

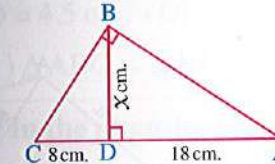
(1)



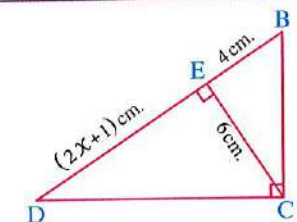
(2)



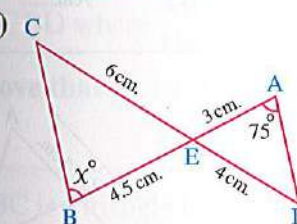
(3)



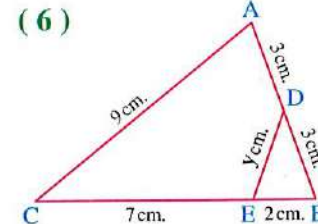
(4)



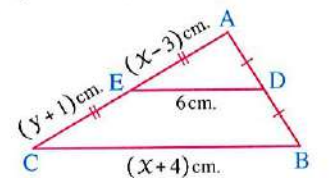
(5)



(6)



(7)

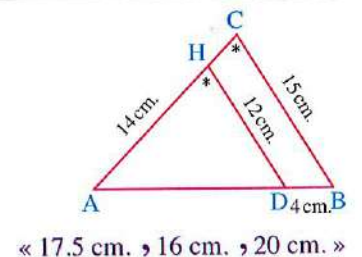


4 In the opposite figure :

$m(\angle AHD) = m(\angle C)$, $AH = 14$ cm, $HD = 12$ cm.

, $CB = 15$ cm, $DB = 4$ cm.

Find the length of each of : \overline{AC} , \overline{AD} , \overline{AB}

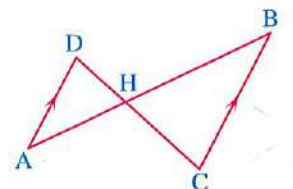


« 17.5 cm, 16 cm, 20 cm. »

5 In the opposite figure :

$\overline{DA} \parallel \overline{CB}$ Prove that :

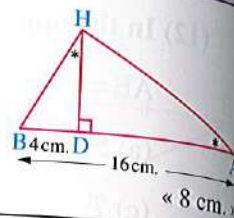
- (1) $\triangle AHD \sim \triangle BHC$
(2) $AH \times HC = DH \times HB$



6 In the opposite figure :

ABH is a triangle, $\overline{HD} \perp \overline{AB}$, $m(\angle A) = m(\angle BHD)$,
 $AB = 16$ cm, $BD = 4$ cm.

Calculate the length of : \overline{BH}

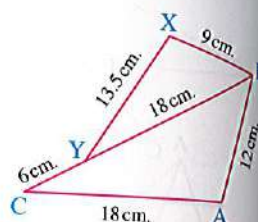
**7** ABC is a triangle, the lengths of its sides \overline{AB} , \overline{BC} and \overline{CA} respectively are 3 cm, 4.5 cm, and 6 cm, DEF is another triangle, the lengths of its sides \overline{DE} , \overline{EF} and \overline{FD} respectively are 6 cm, 4 cm and 8 cm. Prove that the two triangles are similar, then write them in the same order of corresponding vertices.**8 In the opposite figure :**

B, Y and C are collinear.

Prove that :

(1) $\triangle XBY \sim \triangle ABC$

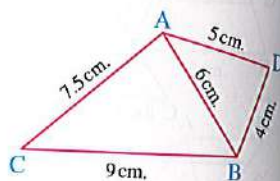
(2) \overline{BC} bisects $\angle ABX$

**9** In the opposite figure :

ABC is a triangle in which : $AB = 6$ cm, $BC = 9$ cm, $AC = 7.5$ cm, D is a point outside the triangle ABC where $DB = 4$ cm, $DA = 5$ cm. Prove that :

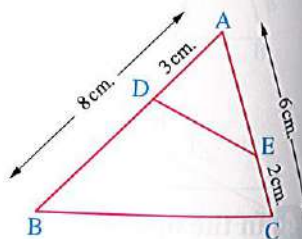
(1) $\triangle ABC \sim \triangle DBA$

(2) \overline{BA} bisects $\angle DBC$

**10 In the opposite figure :**

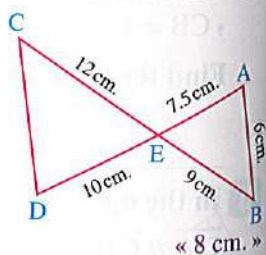
ABC is a triangle in which $AB = 8$ cm, $AC = 6$ cm, $D \in \overline{AB}$, where $AD = 3$ cm, $E \in \overline{AC}$, where $EC = 2$ cm.

Prove that : $\triangle AED \sim \triangle ABC$

**11 In the opposite figure :**

$\overline{AD} \cap \overline{BC} = \{E\}$, $AE = 7.5$ cm, $EC = 12$ cm, $BE = 9$ cm, $ED = 10$ cm, $AB = 6$ cm.

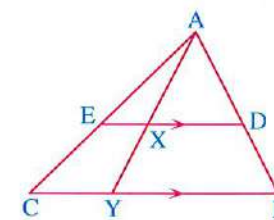
Prove that : $\triangle ABE \sim \triangle DCE$, then find the length of : \overline{CD}

**12** In $\triangle ABC$, $AC > AB$, $M \in \overline{AC}$ where $m(\angle ABM) = m(\angle C)$
 Prove that : $(AB)^2 = AM \times AC$ **13** In the opposite figure :

ABC is a triangle, $D \in \overline{AB}$, $\overline{DE} \parallel \overline{BC}$ and intersects \overline{AC} at E, \overline{AX} is drawn to intersect \overline{DE} and \overline{BC} at X and Y respectively

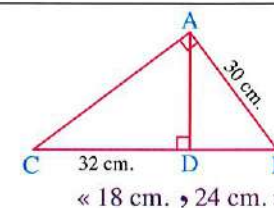
(1) State three pairs of similar triangles.

(2) Prove that : $\frac{DX}{BY} = \frac{XE}{YC} = \frac{DE}{BC}$

**14 In the opposite figure :**

ABC is a right-angled triangle at A, $\overline{AD} \perp \overline{BC}$, $AB = 30$ cm, $DC = 32$ cm.

Calculate the length of each of : \overline{BD} , \overline{AD}

**15 In the opposite figure :**

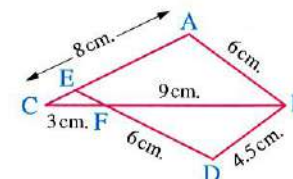
$\overline{BC} \cap \overline{DE} = \{F\}$, $AB = 6$ cm,

$BC = 12$ cm, $AC = 8$ cm, $FC = 3$ cm,

$BD = 4.5$ cm, $DF = 6$ cm. Prove that :

(1) $\triangle ABC \sim \triangle DBF$

(2) $\triangle EFC$ is isosceles.

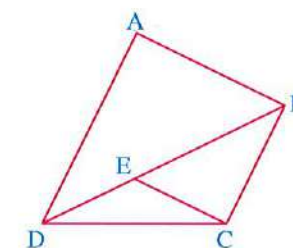
**16** In the opposite figure :

ABCD is a quadrilateral,

$E \in \overline{BD}$ where $\frac{AB}{DA} = \frac{CE}{BC}$, $\frac{BD}{DA} = \frac{EB}{BC}$

Prove that : (1) $\overline{AD} \parallel \overline{BC}$

(2) $\overline{AB} \parallel \overline{CE}$

**17** ABC is a triangle in which : $AB = 4$ cm, $AC = 3$ cm, $D \in \overline{BA}$ such that $AD = 4.5$ cm, $E \in \overline{CA}$ where $AE = 6$ cm.

Prove that : BCDE is a cyclic quadrilateral.

18 ABC is a triangle, $AB = 8$ cm, $AC = 10$ cm, $BC = 12$ cm, $E \in \overline{AB}$ where $AE = 2$ cm, $D \in \overline{BC}$ where $BD = 4$ cm. Prove that :

(1) $\triangle BDE \sim \triangle BAC$ and deduce the length of \overline{DE}

« 5 cm. »

(2) The figure ACDE is a cyclic quadrilateral.

19 XYZ is a right-angled triangle at X, draw $\overline{XL} \perp \overline{YZ}$ and intersects it at L

Prove that : $\frac{(XY)^2}{(XZ)^2} = \frac{YL}{LZ}$

If $XY = 12$ cm, and $XZ = 16$ cm, calculate the length of each of : \overline{YL} , \overline{XL}

« 7.2 cm, 9.6 cm. »

20 Choose the correct answer from those given :

(1) In the opposite figure :

YC = cm.

- (a) 9 (b) 10
(c) 11 (d) 12

(2) In the opposite figure :

If \overrightarrow{DA} , \overrightarrow{DB} are tangents to the circle at A and B respectively, $DA = DB = 8$ cm., $BC = 2$ cm., then $AC =$ cm.

- (a) 3 (b) 4 (c) 5 (d) 6

(3) In the opposite figure :

EC = cm.

- (a) 3 (b) 4
(c) $2\sqrt{5}$ (d) 5

(4) In the opposite figure :

If the perimeter of $\triangle DXY = 8$ cm., then the perimeter of $\triangle ABC =$ cm.

- (a) 18 (b) 24
(c) 36 (d) 48

(5) In the opposite figure :

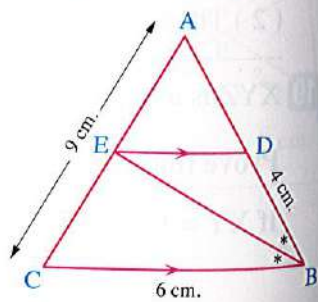
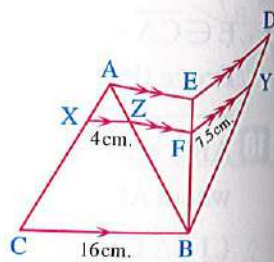
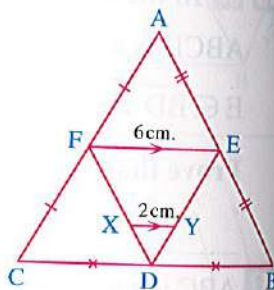
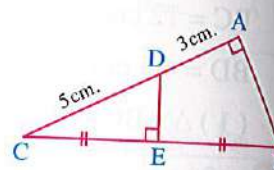
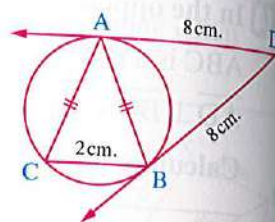
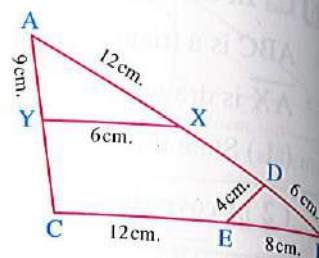
DE = cm.

- (a) 8 (b) 10
(c) 12 (d) 15

(6) In the opposite figure :

If $AC = 9$ cm., $BD = 4$ cm., $BC = 6$ cm., then the perimeter of $\triangle ADE =$ cm.

- (a) 18 (b) 16
(c) 14 (d) 12

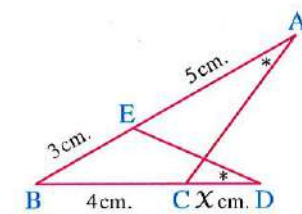


(7) In the opposite figure :

If $m(\angle A) = m(\angle D)$

, then $X =$

- (a) 5 (b) 4
(c) 3 (d) 2



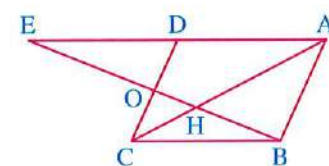
21 In the opposite figure :

ABCD is a parallelogram, $O \in \overline{DC}$,

\overline{BO} is drawn intersecting \overline{AC} at H,

and intersecting \overline{AD} at E

Prove that : (1) $\triangle AHE \sim \triangle CHB$ (2) $(HB)^2 = HE \times HO$



22 \overline{AB} and \overline{DC} are two chords in a circle, $\overline{AB} \cap \overline{CD} = \{E\}$, where E lies outside the circle, $AB = 4$ cm., $DC = 7$ cm. and $BE = 6$ cm.

Prove that : $\triangle ADE \sim \triangle CBE$, then find the length of : \overline{CE}

« 12 cm. »

23 \overline{AB} is a diameter in a circle, C is a point belonging to the circle, \overline{AC} is drawn intersecting the tangent to the circle at B at D

Prove that : $(BC)^2 = CA \times CD$

24 ABC is a right-angled triangle at A, $\overline{AD} \perp \overline{BC}$ to intersect it at D

If $\frac{BD}{DC} = \frac{1}{2}$ and $AD = 6\sqrt{2}$ cm.

, find the length of each of : \overline{BD} , \overline{AB} and \overline{AC}

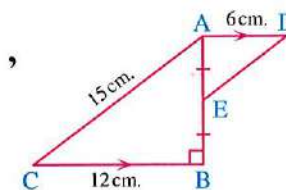
« 6 cm., $6\sqrt{3}$ cm., $6\sqrt{6}$ cm. »

25 In the opposite figure :

$\triangle ABC$ is a right-angled triangle at B, $AC = 15$ cm., $BC = 12$ cm.,

E is the midpoint of \overline{AB} , $\overline{AD} \parallel \overline{BC}$, where $AD = 6$ cm.

Prove that : $\triangle ABC \sim \triangle EAD$ and deduce that $\overline{AC} \parallel \overline{DE}$



26 In the opposite figure :

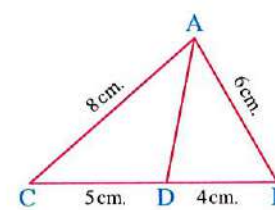
ABC is a triangle in which : $D \in \overline{BC}$ where $BD = 4$ cm.,

$DC = 5$ cm. If $AB = 6$ cm., $AC = 8$ cm.

(1) Prove that : $\triangle ABC \sim \triangle DBA$

(2) Find the length of : \overline{AD}

(3) Prove that : \overline{AB} is a tangent segment for the circle passing through the vertices of $\triangle ADC$



« $5\frac{1}{3}$ cm. »

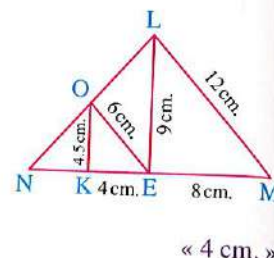
27 In the opposite figure :

LMN is a triangle, $E \in \overline{MN}$, $K \in \overline{MN}$

, $O \in \overline{LN}$, $LM = 12$ cm, $ME = 8$ cm,

$LE = 9$ cm, $EO = 6$ cm, $EK = 4$ cm, $KO = 4.5$ cm.

Prove that : $\overline{OK} \parallel \overline{LE}$, $\overline{EO} \parallel \overline{ML}$, then find the length of \overline{NK}



« 4 cm. »

28 XYZ, LMN are two triangles having equal measures of corresponding angles, $YZ = 8$ cm,

, $MN = 12$ cm, \overline{XD} is drawn $\perp \overline{YZ}$ to intersect it at D, and \overline{LH} is drawn $\perp \overline{MN}$ to intersect it at H

If $DX = 7$ cm, find the length of : \overline{LH}

« 10.5 cm. »

29 ABC and DEF are two similar triangles, $\overline{AX} \perp \overline{BC}$ to intersect it at X, $\overline{DY} \perp \overline{EF}$ to intersect it at Y. Prove that : $BX \times YF = CX \times YE$ 30 ABC is a triangle, $AB = 9$ cm, $BC = 12$ cm, $CA = 15$ cm, $D \in \overline{BC}$ such that : $BD = \frac{1}{4} BC$, $\overline{DH} \perp \overline{BC}$ to intersect \overline{AC} at H

Find the area of the shape : ABDH

« $23\frac{5}{8}$ cm² »31 ABC is a right-angled triangle at A, $D \in \overline{BC}$ where $\frac{DB}{AB} = \frac{BA}{BC}$

Prove that : (1) $\triangle ABC \sim \triangle DBA$ (2) $\overline{AD} \perp \overline{BC}$

32 If $\triangle ABC \sim \triangle DEF$ and X is the midpoint of \overline{BC} , Y is the midpoint of \overline{EF} , prove that : $\triangle ABX \sim \triangle DEY$ 33 ABCD is a quadrilateral inscribed in a circle, its diagonals \overline{AC} , \overline{BD} intersect at E, If $\frac{BA}{AE} = \frac{BD}{DC}$, prove that :

(1) $\triangle ABE \sim \triangle DBC$

(2) \overline{BD} bisects $\angle ABC$

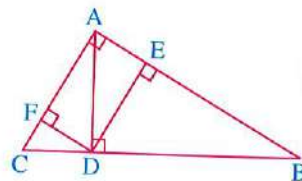
34 In the opposite figure :

ABC is a right-angled triangle at A

, $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{AC}$

Prove that : (1) $\triangle ADE \sim \triangle CDF$

(2) Area of the rectangle AEDF = $\sqrt{AE \times EB \times AF \times FC}$

35 ABCD is a rectangle, draw $\overline{DF} \perp \overline{AC}$ to intersect \overline{AC} in E and \overline{BC} in F. Prove that : The area of the rectangle ABCD = $\sqrt{AE \times AC \times DE \times DF}$ 36 ABCD is a trapezium in which : $\overline{AD} \parallel \overline{BC}$, its two diagonals \overline{AC} , \overline{BD} intersect at M. Prove that : $MA \times MB = MC \times MD$, and if $AD = 9$ cm, $BC = 12$ cm, $AC = 14$ cm, calculate the length of : \overline{MA}

« 6 cm. »

37 ABC is a triangle, $D \in \overline{BC}$, \overline{AD} is drawn and point H is assumed on it, then \overline{HX} is drawn $\parallel \overline{AB}$ to intersect \overline{BD} at X, and \overline{HY} is drawn $\parallel \overline{AC}$ to intersect \overline{DC} at Y

Prove that : (1) $\triangle ABC \sim \triangle HXY$

(2) $XY \times AD = BC \times DH$

38 \overline{AB} is a diameter in circle M, $C \in \overline{AB}$ lying outside the circle, \overline{CD} is drawn tangent to the circle at point D, then $\overline{DH} \perp \overline{AB}$ to intersect it at H

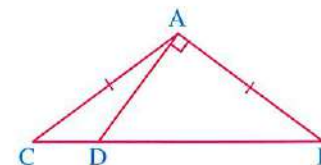
Prove that : $(CD)^2 = CH \times CM = CB \times CA$

39 In the opposite figure :

ABC is an obtuse-angled triangle at A,

$AB = AC$, $\overline{AD} \perp \overline{BC}$ and intersects \overline{BC} at D

Prove that : $2(AB)^2 = BD \times BC$

40 ABCD is a trapezium, $\overline{AD} \parallel \overline{BC}$, $m(\angle A) = 90^\circ$, $E \in \overline{BD}$

, where $AB \times EC = DE \times BD$, $CD \times BD = DA \times EC$

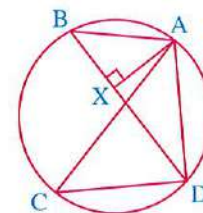
Prove that : $(BC)^2 = (AB)^2 + (AD)^2 + (CD)^2$

41 In the opposite figure :

$\overline{AX} \perp \overline{BD}$, $\frac{BX}{CD} = \frac{BA}{CA}$. Prove that :

(1) $\triangle BXA \sim \triangle CDA$

(2) \overline{AC} is a diameter in the circle.

42 ABC is a triangle in which $AB = AC$, $E \in \overline{BC}$, $E \notin \overline{BC}$, $D \in \overline{CB}$, $D \notin \overline{CB}$ where $(AB)^2 = DB \times CE$. Prove that : $\triangle ABD \sim \triangle ECA$

Problems that measure high standard levels of thinking

43 Choose the correct answer from those given :

(1) In the opposite figure :

$$\text{If } \frac{x-y}{x+y} = \frac{2}{7}$$

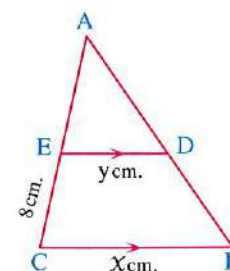
, then $AE = \dots\dots\dots$ cm.

(a) 16

(b) 15

(c) 12

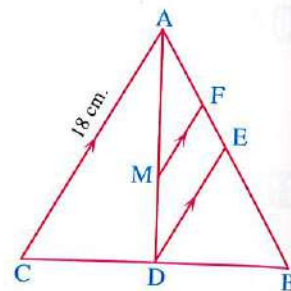
(d) 10



(2) In the opposite figure :

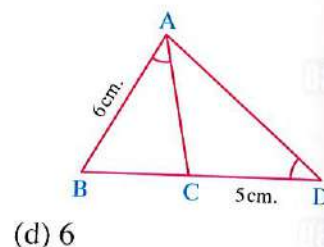
If M is the point of intersection
of medians in $\triangle ABC$
, then the length of \overline{FM} = cm.

- (a) 4 (b) 5
(c) 6 (d) 8

**(3) In the opposite figure :**

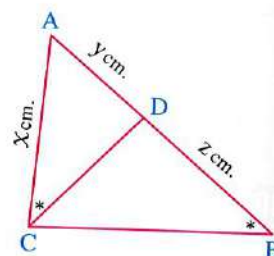
$C \in \overline{BD}$, $m(\angle D) = m(\angle BAC)$
, $AB = 6$ cm. , $CD = 5$ cm.
, then BC = cm.

- (a) 3 (b) 4 (c) 5

**(4) In the opposite figure :**

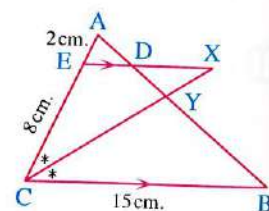
If $x^2 - y^2 = 16$
, then $y \times z$ = cm^2

- (a) 4 (b) 8
(c) 12 (d) 16

**(5) In the opposite figure :**

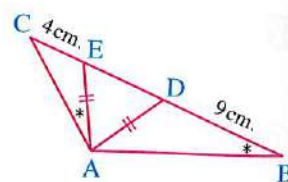
If \overline{CX} bisects $\angle ACB$, $\overline{XD} \parallel \overline{BC}$
, then XD = cm.

- (a) 3 (b) 4
(c) 5 (d) 6

**(6) In the opposite figure :**

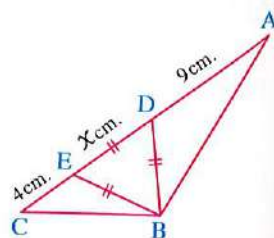
AD = cm.

- (a) 10 (b) 9
(c) 8 (d) 6

**(7) In the opposite figure :**

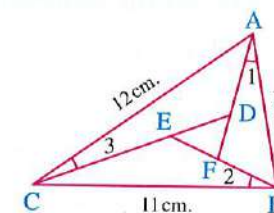
If $m(\angle ABC) = 120^\circ$
, $\triangle BDE$ is an equilateral triangle
, then x = cm.

- (a) 5 (b) 6
(c) 7 (d) 8

**(8) In the opposite figure :**

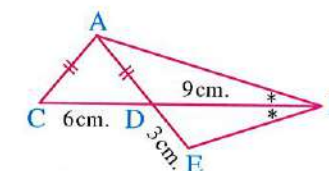
If $m(\angle 1) = m(\angle 2) = m(\angle 3)$
, then $DE : EF : FD$ = : :

- (a) 7 : 11 : 12 (b) 12 : 11 : 7
(c) 12 : 7 : 11 (d) 11 : 12 : 7

**(9) In the opposite figure :**

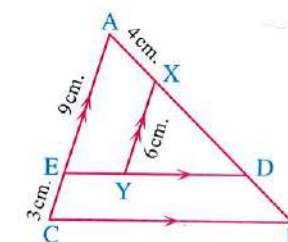
If \overline{BD} bisects $\angle ABE$, $BD = 9$ cm. , $DC = 6$ cm.
, $DE = 3$ cm. , then the perimeter of $\triangle ADC$ = cm.

- (a) 12 (b) 14
(c) 16 (d) 18

**(10) In the opposite figure :**

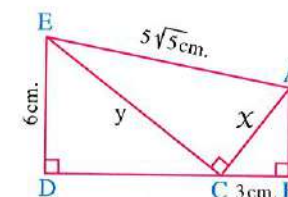
$\overline{XY} \parallel \overline{AC}$, $\overline{DE} \parallel \overline{BC}$
, then DB = cm.

- (a) 2 (b) 3
(c) 4 (d) 5

**(11) In the opposite figure :**

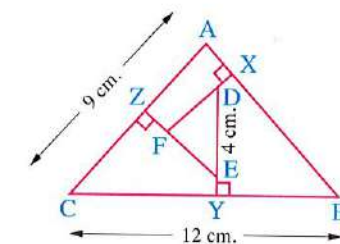
$x + y$ = cm.

- (a) 12 (b) 15
(c) 18 (d) 21

**(12) In the opposite figure :**

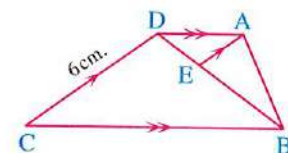
If $\overline{FX} \perp \overline{AB}$, $\overline{DY} \perp \overline{BC}$, $\overline{EZ} \perp \overline{AC}$
, $AC = 9$ cm. , $BC = 12$ cm. , $DE = 4$ cm.
, then EF = cm.

- (a) 2 (b) 3
(c) 5 (d) 6

**(13) In the opposite figure :**

If $BE = 2 ED$
, then AE = cm.

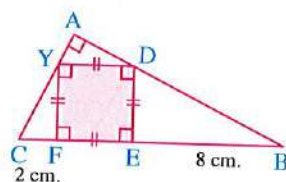
- (a) 1 (b) 2
(c) 3 (d) 4



(14) In the opposite figure :

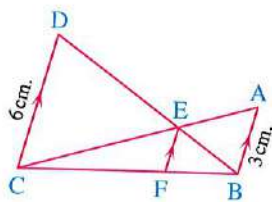
If $\triangle ABC$ is a right-angled triangle at A
 , $DEFY$ is a square , $BE = 8$ cm. , $FC = 2$ cm.
 , then the area of the square $DEFY = \dots\dots\dots \text{cm}^2$

- (a) 4 (b) 16
 (c) 20 (d) 36

**(15) In the opposite figure :**

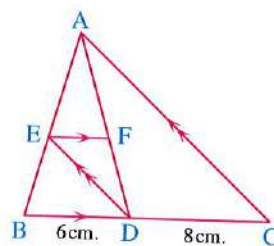
If $\overline{AB} \parallel \overline{EF} \parallel \overline{CD}$
 , then $EF = \dots\dots\dots$ cm.

- (a) 2.5 (b) 2
 (c) 1.5 (d) 1

**(16) In the opposite figure :**

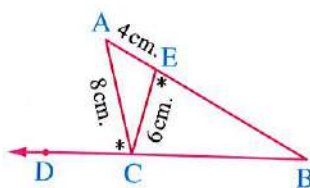
$\overline{EF} \parallel \overline{BC}$, $\overline{DE} \parallel \overline{CA}$
 If $BD = 6$ cm. , $DC = 8$ cm.
 , then $EF = \dots\dots\dots$ cm.

- (a) $\frac{12}{7}$ (b) $\frac{18}{7}$
 (c) $\frac{24}{7}$ (d) $\frac{28}{7}$

**(17) In the opposite figure :**

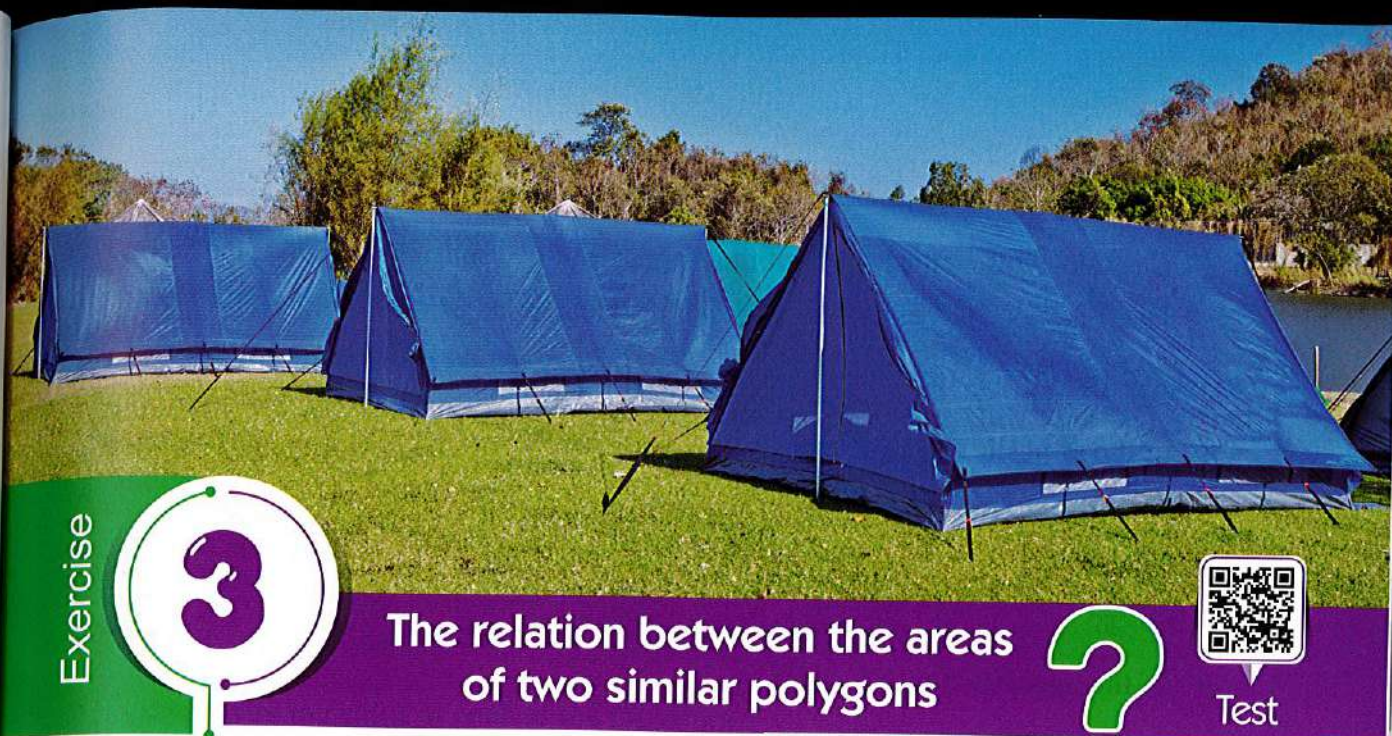
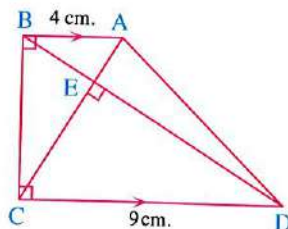
If $m(\angle ACD) = m(\angle BEC)$
 , then $BE + BC = \dots\dots\dots$ cm.

- (a) 16 (b) 18
 (c) 20 (d) 24

**(18) In the opposite figure :**

$ABCD$ is a trapezium , $m(\angle ABC) = m(\angle DCB) = 90^\circ$
 , $\overline{AC} \perp \overline{BD}$, then the area of the trapezium
 $ABCD = \dots\dots\dots \text{cm}^2$

- (a) 13 (b) 26
 (c) 39 (d) 60



Exercise

3

The relation between the areas
of two similar polygons

Test
yourself

From the school book

1 Choose the correct answer from those given :

(1) If the lengths of two corresponding sides in two similar polygons are 7 cm.
 and 11 cm. , then the ratio between their perimeters is

- (a) $\frac{49}{121}$ (b) $\frac{7}{18}$ (c) $\frac{7}{11}$ (d) $\frac{11}{18}$

(2) If $\triangle ABC \sim \triangle XYZ$, $AB = 3 XY$, then $\frac{a(\triangle XYZ)}{a(\triangle ABC)} = \dots\dots\dots$

- (a) 3 (b) 9 (c) $\frac{1}{3}$ (d) $\frac{1}{9}$

(3) If the ratio between the areas of two similar polygons is 9 : 49 , then the ratio
 between the lengths of their two corresponding sides is

- (a) 3 : 7 (b) 9 : 49 (c) 3 : 10 (d) 10 : 3

(4) If $\triangle ABC \sim \triangle DEF$, $a(\triangle ABC) = 9 a(\triangle DEF)$ and $DE = 4$ cm. , then $AB = \dots\dots\dots$ cm.

- (a) $\frac{4}{3}$ (b) 12 (c) 9 (d) 36

(5) The ratio between the perimeters of two similar polygons is 4 : 9 , so the ratio
 between their areas is

- (a) 4 : 9 (b) 9 : 4 (c) 2 : 3 (d) 16 : 81

(6) The ratio between the areas of two similar polygons is 9 : 25 and the length of one
 side of the smaller one is 3 cm. , so the length of the corresponding side in the greater
 one is

- (a) $\frac{25}{3}$ (b) $\frac{9}{5}$ (c) 75 (d) 5

(7) The ratio between the lengths of the diagonals of two squares is 2 : 5, if the area of the smaller one is 4 cm^2 , so the area of the greater one is cm^2

- (a) 25 (b) 16 (c) 10 (d) 20

(8) In the opposite figure :

$$\frac{a(\Delta ABE)}{a(\Delta CDE)} = \dots\dots\dots$$

- (a) $\frac{9}{49}$ (b) $\frac{25}{49}$ (c) $\frac{9}{25}$

(9) In the opposite figure :

$$\frac{\text{Area}(\Delta XAB)}{\text{Area}(\Delta XYZ)} = \dots\dots\dots$$

- (a) $\frac{3}{5}$ (b) $\frac{5}{16}$
(c) $\frac{9}{25}$ (d) $\frac{4}{5}$

(10) In the opposite figure :

If $AX : XB = 5 : 3$, $a(\Delta ABC) = 25.6 \text{ cm}^2$

, then $a(\Delta AXY) = \dots\dots\dots \text{cm}^2$

- (a) 10 (b) 16
(c) 41 (d) 65.5

(11) In the opposite figure :

If $\overline{BE} \parallel \overline{DC}$

, then $\frac{\text{The area of } \Delta ABE}{\text{The area of trapezium BCDE}} = \dots\dots\dots$

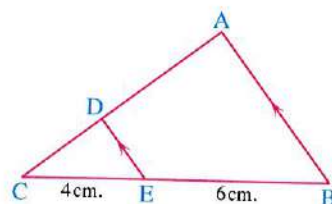
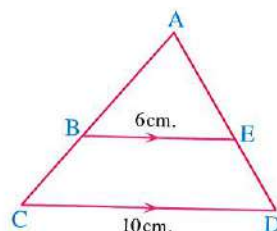
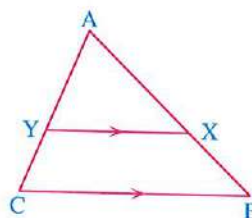
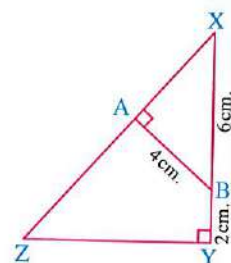
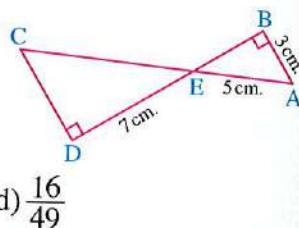
- (a) $\frac{25}{81}$ (b) $\frac{3}{5}$
(c) $\frac{9}{16}$ (d) $\frac{9}{25}$

(12) In the opposite figure :

If the area of the figure ABED = 42 cm^2

, then the area of $\Delta CED = \dots\dots\dots \text{cm}^2$

- (a) 8 (b) 12
(c) 16 (d) 20



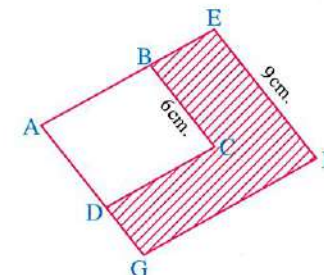
(13) In the opposite figure :

If the polygon ABCD ~ the polygon AEFG

and the area of the polygon ABCD = 32 cm^2

, then the shaded area = cm^2

- (a) 72 (b) 48
(c) 40 (d) 16



2 If the polygon ABCD ~ the polygon A'B'C'D', $\frac{AB}{A'B'} = \frac{1}{3}$

, then write the value of each of the following :

a (the polygon ABCD) and perimeter of (ABCD)
a (the polygon A'B'C'D') and perimeter of (A'B'C'D')

3 If the lengths of two corresponding sides in two similar polygons are 12 cm. , 16 cm. and the area of the smaller polygon = 135 cm^2 , then find the area of the greater polygon.

« 240 cm^2 »

4 The ratio between the lengths of two corresponding sides in two similar triangles is 2 : 5

If the area of the smaller one is 16 cm^2 , find the area of the greater triangle.

« 100 cm^2 »

5 The areas of two similar polygons are 100 cm^2 , 64 cm^2 . If the perimeter of the first is 60 cm.

, find the perimeter of the other polygon.

« 48 cm. »

6 The ratio between the two perimeters of two similar triangles is 3 : 2 and the sum of their areas is 130 cm^2 . Find the area of each of them.

« 90 cm^2 , 40 cm^2 »

7 The ratio between the lengths of two corresponding sides in two similar polygons is 1 : 3

Let the difference between their areas be 32 cm^2 , so find the area of each.

« 4 cm^2 , 36 cm^2 »

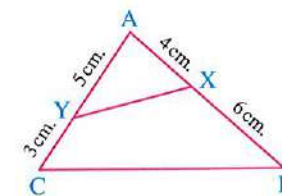
8 In the opposite figure :

ABC is a triangle in which :

$AX = 4 \text{ cm}$, $XB = 6 \text{ cm}$,

$AY = 5 \text{ cm}$, $YC = 3 \text{ cm}$.

Find : $\frac{a(\Delta AXY)}{a(\Delta ACB)}$

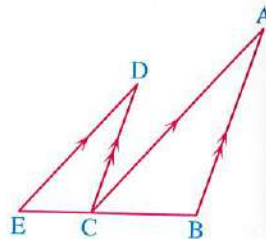


« $\frac{1}{4}$ »

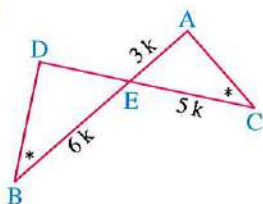
9 In the opposite figure :

If $\overline{AB} \parallel \overline{DC}$, $\overline{AC} \parallel \overline{DE}$,

$$AB = \frac{3}{2} DC$$

, area of $\triangle DCE = 16 \text{ cm}^2$, find the area of : $\triangle ABC$ « 36 cm^2 »10 Study each of the following figures, where k is the constant of proportion, then find the requirement under each figure :

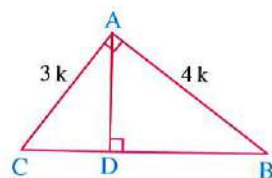
(1)



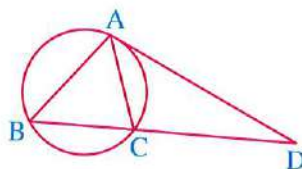
$$\overline{AB} \cap \overline{CD} = \{E\}$$

, a ($\triangle ACE$) = 900 cm^2 Find area of $\triangle DEB$

(2)

 $m(\angle BAC) = 90^\circ$, $\overline{AD} \perp \overline{BC}$, a ($\triangle ADC$) = 180 cm^2 Find area of $\triangle ABC$ 11 ABC is a triangle, $D \in \overline{AB}$ where $AD = 2 BD$, $E \in \overline{AC}$ where $\overline{DE} \parallel \overline{BC}$ If the area of $\triangle ADE = 60 \text{ cm}^2$, find the area of the trapezium DBCE« 75 cm^2 »12 ABC is a triangle, $AB = 8 \text{ cm}$, $AC = 6 \text{ cm}$, $D \in \overline{AB}$ where $AD = 3 \text{ cm}$,, $E \in \overline{AC}$ where $EC = 2 \text{ cm}$. Find : $\frac{a(\triangle ADE)}{a(\text{figure DBCE})}$ « $\frac{1}{3}$ »13 ABCD, $\hat{A}\hat{B}\hat{C}\hat{D}$ are two similar polygons whose diagonals intersect at X, Y respectivelyProve that : $\frac{a(\text{the polygon ABCD})}{a(\text{the polygon } \hat{A}\hat{B}\hat{C}\hat{D})} = \frac{(BX)^2}{(BY)^2}$

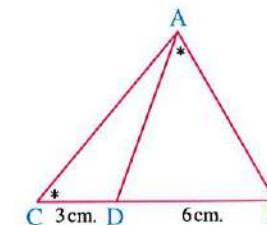
14 In the opposite figure :

 \overline{AD} is a tangent segmentto the circumcircle of $\triangle ABC$, $2 AB = 3 AC$ Find : $\frac{a(\triangle ACD)}{a(\triangle ACB)}$ « $\frac{4}{5}$ »

15 In the opposite figure :

ABC is a triangle where $BC = 9 \text{ cm}$.and $D \in \overline{BC}$ where $BD = 6 \text{ cm}$.If $m(\angle BAD) = m(\angle C)$,then prove that : $\triangle ABC \sim \triangle DBA$ and find the length of : \overline{AB}

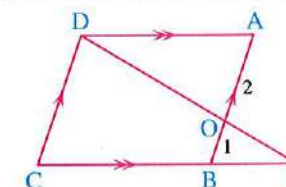
Find also : The ratio between

the area of $\triangle ABC$ and $\triangle DBA$ « $3\sqrt{6} \text{ cm}$, $3:2$ »

16 In the opposite figure :

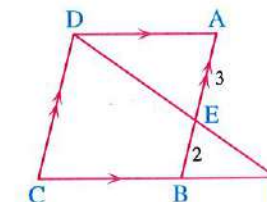
ABCD is a parallelogram, $\frac{BO}{AO} = \frac{1}{2}$, a ($\triangle BEO$) = 9 cm^2

Find : The area of the parallelogram ABCD

« 108 cm^2 »

17 In the opposite figure :

ABCD is a parallelogram

, $E \in \overline{AB}$ where $\frac{AE}{EB} = \frac{3}{2}$, $\overline{DE} \cap \overline{CB} = \{F\}$ (1) Prove that : $\triangle DCF \sim \triangle EAD$ (2) Find : $\frac{a(\triangle DCF)}{a(\triangle EAD)}$ « $\frac{25}{9}$ »18 ABCD is a parallelogram, $X \in \overline{AB}$, $X \notin \overline{AB}$ where $BX = 2 AB$, $Y \in \overline{CB}$, $Y \notin \overline{CB}$ where $BY = 2 BC$, the parallelogram BXZY is drawn.Prove that : $\frac{a(\text{parallelogram ABCD})}{a(\text{parallelogram XBYZ})} = \frac{1}{4}$ 19 ABCD, XYZL are two similar polygons. If M is the midpoint of \overline{BC} and N is the midpoint of \overline{YZ} , prove that : $a(\text{polygon ABCD}) : a(\text{polygon XYZL}) = (MD)^2 : (NL)^2$ 20 \overline{AB} , \overline{CD} are two non intersecting chords of circle MIf $\overline{AB} \cap \overline{CD} = \{E\}$, $AC = 3 BD$, find : $\frac{a(\triangle EBD)}{a(\triangle ECA)}$ « $\frac{1}{9}$ »

- 21 M, N are two touching externally circles at A, the two secants from A are drawn to intersect the circle M at B, D and intersect the circle N at C, E

Prove that : $\frac{a(\Delta ABD)}{a(\Delta ACE)} = \frac{(BD)^2}{(CE)^2}$

- 22 ABC is a triangle inscribed inside a circle, draw \overline{AD} to bisect $\angle A$ and intersect \overline{BC} at D and the circle at E

Prove that : $a(\Delta ABE) : a(\Delta ADC) : a(\Delta BDE) = (EB)^2 : (CD)^2 : (ED)^2$

- 23 If $\Delta ABC \sim \Delta XYZ$, \overline{AD} , \overline{XL} are their corresponding heights, **prove that :** $BC \times XL = AD \times YZ$

- 24 **Prove that :** the ratio between the areas of the two similar triangles equals the square of the ratio between :

(1) Two corresponding heights in them.

(2) The lengths of two corresponding medians in them.

- 25 ABC is a right-angled triangle at B. The equilateral triangles ABX, BCY, ACZ are drawn. **Prove that :** $a(\Delta ABX) + a(\Delta BCY) = a(\Delta ACZ)$

- 26 ABC is an inscribed triangle in a circle where $\frac{AB}{BC} = \frac{4}{3}$, from B a tangent is drawn to the circle to intersect \overline{AC} at E

Prove that : $\frac{a(\Delta ABC)}{a(\Delta ABE)} = \frac{7}{16}$

- 27 ABCD is a trapezium in which $\overline{AD} \parallel \overline{BC}$. Draw $\overline{XY} \parallel \overline{AD}$ to intersect \overline{AB} at X and \overline{CD} at Y such that the trapezium is divided into two similar polygons AXYD and XBCY

Prove that : $\frac{a(\text{polygon AXYD})}{a(\text{polygon XBCY})} = \frac{a(\Delta ABD)}{a(\Delta BDC)}$

- 28 ΔABC is right-angled at A, $\overline{AD} \perp \overline{BC}$ intersecting it at D. The two equilateral triangles ABE, CAF are drawn outside the triangle ABC

Prove that : (1) The polygon ADBE \sim the polygon CDAF

(2) $\frac{a(\text{the polygon ADBE})}{a(\text{the polygon CDAF})} = \frac{BD}{CD}$

- 29 ABC is a right-angled triangle at B, $\overline{BD} \perp \overline{AC}$ to intersect it at D. The squares AXYB, BMNC are drawn on \overline{AB} , \overline{BC} respectively outside the triangle ABC

(1) **Prove that :** The polygon DAXYB \sim the polygon DBMNC

(2) If $AB = 6$ cm, $AC = 10$ cm.

find : the ratio between areas of the two polygons.

$\ll \frac{9}{16} \gg$

- 30 ABC is a triangle in which \overline{AB} , \overline{BC} , \overline{AC} are corresponding sides to three similar polygons X, Y, Z drawn outside the triangle respectively. If the area of the polygon $X = 40$ cm², the area of $Y = 85$ cm², the area of $Z = 125$ cm², **prove that :** ΔABC is a right-angled triangle.

- 31 ABCD is a quadrilateral, $E \in \overline{BD}$, draw $\overline{EF} \parallel \overline{DA}$ to intersect \overline{AB} at F, draw $\overline{EM} \parallel \overline{DC}$ and intersects \overline{BC} at M
Prove that : $a(\text{the polygon BMEF}) : a(\text{the polygon BCDA}) = \frac{BF \times BM}{BA \times BC}$

- 32 ABCD is a square, \overline{AB} , \overline{BC} , \overline{CD} , \overline{DA} are divided in ratio 1 : 3 by the points X, Y, Z, L respectively.

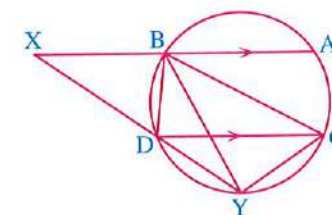
Prove that : (1) XYZL is a square. (2) $\frac{a(\text{the square XYZL})}{a(\text{the square ABCD})} = \frac{5}{8}$

- 33 In the opposite figure :

\overline{AB} , \overline{CD} are two parallel chords

in a circle, $\overline{AB} \cap \overline{YD} = \{X\}$

Prove that : $\frac{a(\Delta DBX)}{a(\Delta CYB)} = \frac{(XB)^2}{(BY)^2}$



Problems that measure high standard levels of thinking

- 34 Choose the correct answer from those given :

(1) In the opposite figure :

If the area of (polygon DYFC) = 40 cm²

, the area of (polygon FEBC) = 32 cm²

, the area of $(\Delta AFY) = 5$ cm²

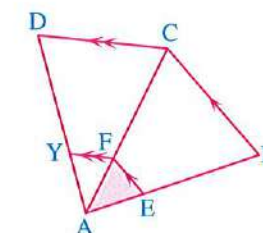
, then the area of $(\Delta AEF) = \dots\dots\dots$ cm²

(a) 3

(b) 4

(c) 5

(d) 6



(2) In the opposite figure :

If the area of $(\triangle AXY) = 40 \text{ cm}^2$

, the area of $(\triangle DZM) = 13 \text{ cm}^2$

, the area of (the polygon XBCY) = 50 cm^2

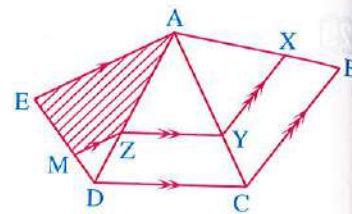
Then the shaded area = cm^2

(a) 77

(b) 92

(c) 104

(d) 112

**(3) In the opposite figure :**

If $AB = 3 AD$, and the area

of $\triangle ADE = 6 \text{ cm}^2$

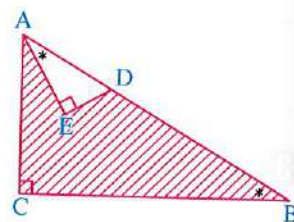
, then the shaded area = cm^2

(a) 12

(b) 24

(c) 48

(d) 96

**(4) In the opposite figure :**

If the area of the polygon DXYE = 30 cm^2

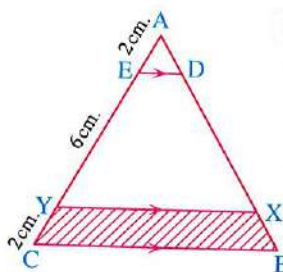
, then the area of the polygon XBCY = cm^2

(a) 12

(b) 16

(c) 18

(d) 20

**(5) In the opposite figure :**

If M is the point of intersection of medians of $\triangle ABC$

, $\overline{MD} \parallel \overline{AB}$ and the area of $\triangle ABC = 36 \text{ cm}^2$

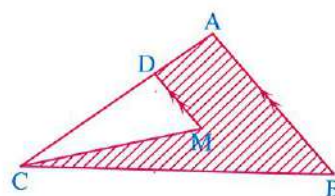
, then the shaded area = cm^2

(a) 27

(b) 28

(c) 32

(d) 33

**(6) In the opposite figure :**

If the area of $\triangle DEF = 6 \text{ cm}^2$

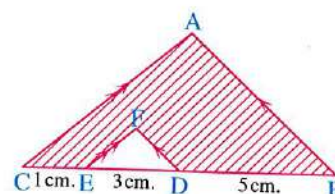
, then the shaded area = cm^2

(a) 27

(b) 36

(c) 48

(d) 54



(7) If $\triangle ABC \sim \triangle DEF$ and $AB = x \text{ cm}$, $DE = (x + 1) \text{ cm}$, the area of $\triangle ABC = (x + 2) \text{ cm}^2$, and the area of $\triangle DEF = (x + 7) \text{ cm}^2$, then the value of $x = \dots\dots\dots$

(a) 4

(b) 3

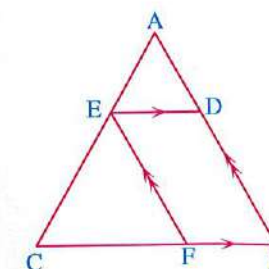
(c) 2

(d) 1

(8) In the opposite figure :

If $\overline{DE} \parallel \overline{BC}$, $\overline{EF} \parallel \overline{AB}$, $\frac{AD}{DB} = \frac{2}{3}$

, then $\frac{\text{Area}(\square DBFE)}{\text{Area}(\triangle ABC)} = \dots\dots\dots$

(a) $\frac{21}{25}$ (b) $\frac{16}{25}$ (c) $\frac{12}{25}$ (d) $\frac{13}{25}$ **(9) In the opposite figure :**

ABCD is a square of side length 6 cm.

, $DE = EF = FC$

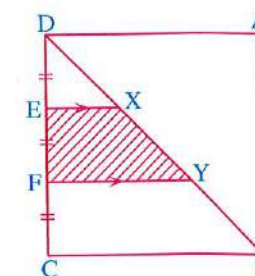
, then the area of (polygon XYFE) = cm^2

(a) 6

(b) 8

(c) 10

(d) 12

**(10) In the opposite figure :**

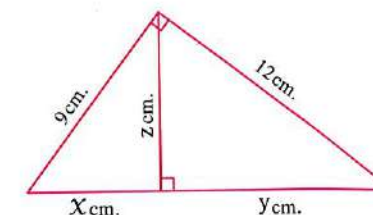
$x + y + z = \dots\dots\dots$

(a) 15

(b) 18.2

(c) 22

(d) 22.2

**(11) In the opposite figure :**

BCDF is a rectangle, the area of $(\triangle ABE) = 2 \text{ cm}^2$

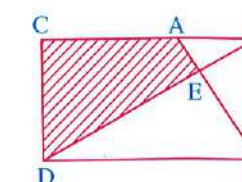
, the area of $(\triangle BEF) = 3 \text{ cm}^2$

, then the shaded area = cm^2

(a) 5

(b) $5\frac{1}{2}$

(c) 6

(d) $7\frac{1}{2}$ 

(12) If the scale factor of similarity of the polygon P_1 to the polygon P_2 is $\frac{2}{3}$ and the scale factor of similarity of the polygon P_3 to the polygon P_2 is $\frac{1}{3}$, which of the following relations is correct ?

(a) $\text{Area}(P_1) + \text{Area}(P_2) = \text{Area}(P_3)$

(b) $\text{Area}(P_1) + \text{Area}(P_3) = \text{Area}(P_2)$

(c) $\sqrt{\text{Area}(P_1)} + \sqrt{\text{Area}(P_2)} = \sqrt{\text{Area}(P_3)}$

(d) $\sqrt{\text{Area}(P_1)} + \sqrt{\text{Area}(P_3)} = \sqrt{\text{Area}(P_2)}$

- 35** \overline{AB} is a diameter in a circle, C belongs to the circle, $X \in \overline{AB}$ where $AX = BC$, draw $\overline{XY} \parallel \overline{BC}$ and intersects \overline{AC} at Y

Prove that : $a(\Delta ABC) : a(\text{the polygon } XBCY) = (AB)^2 : (AC)^2$

- 36** In the opposite figure :

Two intersecting circles at A, B

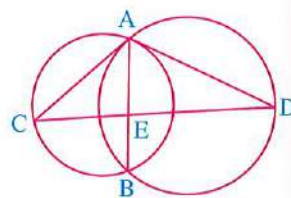
, \overline{AC} is a chord in one of the

two circles and touches the other at A ,

\overline{AD} is a chord in the second circle and touches the first circle at A

If $\overline{AB} \cap \overline{CD} = \{E\}$

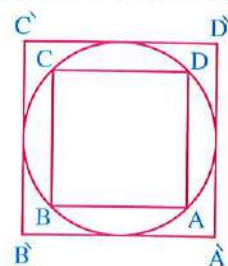
, **prove that :** $\frac{CE}{ED} = \frac{(AC)^2}{(AD)^2}$



- 37** In the opposite figure :

Two squares are drawn, one of them is inside a circle and the other is outside the circle.

Find the ratio between their areas.



« $\frac{1}{2}$ »

Exercise

4

Applications of similarity in the circle

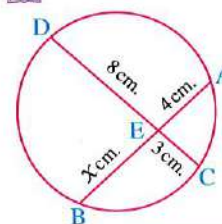


Test
yourself

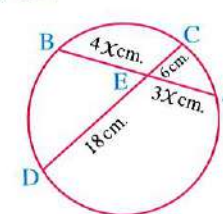
From the school book

- 1** Use the calculator or mental math to find the numerical value of X in each of the following figures :

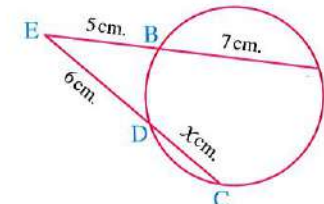
(1)



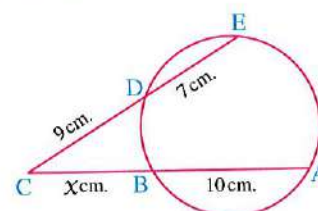
(2)



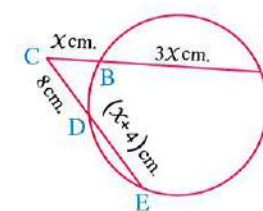
(3)



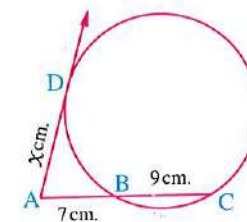
(4)



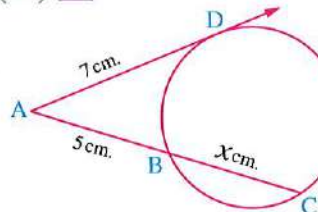
(5)



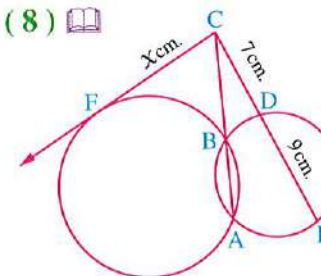
(6)



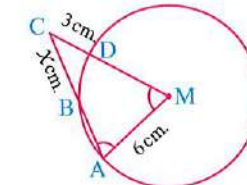
(7)



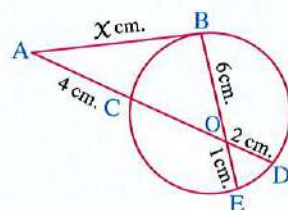
(8)



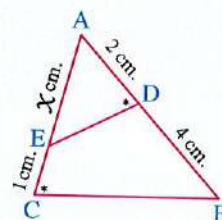
(9)



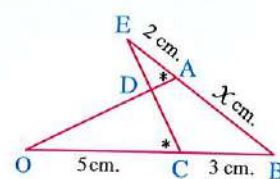
(10)



(11)

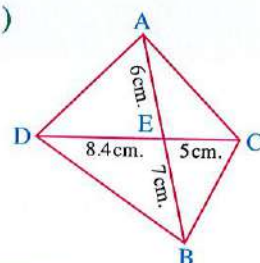


(12)

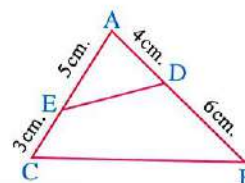


- 2 In which of the following figures, the points A, B, C and D lie on a circle? Explain your answer.

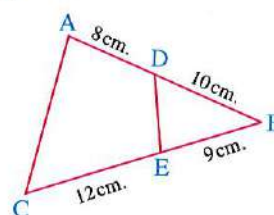
(1)



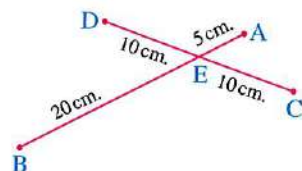
(2)



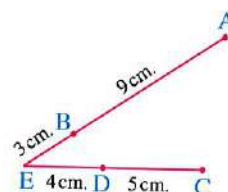
(3)



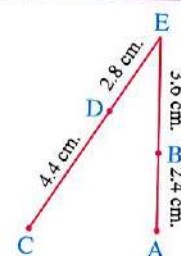
(4)



(5)

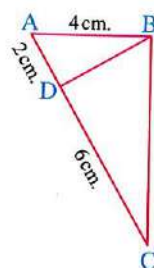


(6)

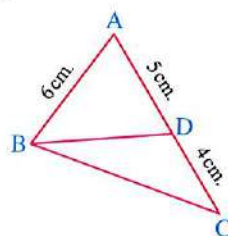


- 3 In which of the following figures, \overline{AB} is a tangent segment to the circle which passes through the points B, C and D?

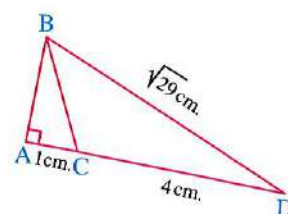
(1)



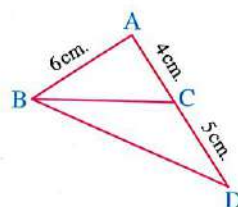
(2)



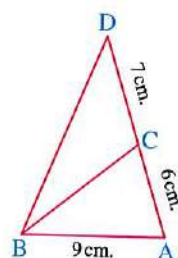
(3)



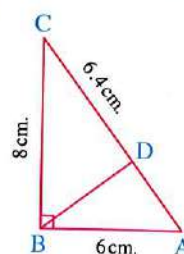
(4)



(5)



(6)

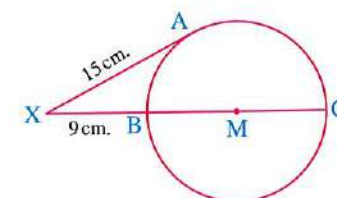


- 4 In the opposite figure :

\overline{XA} is a tangent to the circle M at A

where $XA = 15$ cm. If $XB = 9$ cm.

, calculate the length of the radius of the circle.



« 8 cm. »

- 5 The length of the radius of a circle of center O is 4 cm. Assume a point M such that $MO = 6$ cm. Let \overline{MB} be drawn to intersect the circle at A and B, where $A \in \overline{MB}$

If $MA = 3$ cm. , so find the length of : \overline{AB}

« $3\frac{2}{3}$ cm. »

- 6 \overline{AB} and \overline{CD} are two intersecting chords at E in a circle. If the lengths of \overline{AE} , \overline{BE} , \overline{CE} respectively are 5 cm. , 6 cm. , 11.5 cm. , calculate the lengths of : \overline{ED}

« 7.5 cm. , 4 cm. »

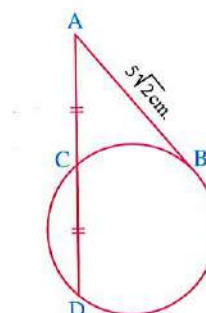
- 7 In the opposite figure :

If \overline{AB} is a tangent segment to the circle at B ,

C is the midpoint of \overline{AD} ,

$AB = 5\sqrt{2}$ cm.

, find the length of : \overline{AD}



« 10 cm. »

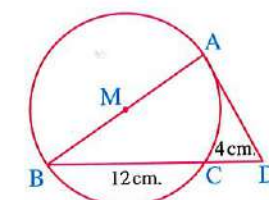
- 8 In the opposite figure :

\overline{AB} is a diameter in the circle M ,

\overline{AD} is a tangent to the circle at A

Find the area of the circle M

« 48π cm.² »

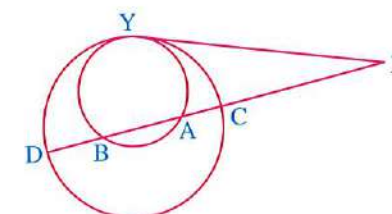


- 9 In the opposite figure :

Two circles are touching internally at point Y ,

\overline{YX} is a common tangent to the two circles.

Prove that : $\frac{XC}{XB} = \frac{XA}{XD}$



(8) In the opposite figure :

A, B, D are three points on a circle whose centre M

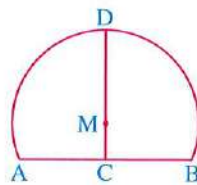
If C is the midpoint of \overline{AB}

, D, M, C are collinear ,

AB = 24 cm. , DC = 18 cm.

, then the radius of the circle = cm.

- (a) 9 (b) 8 (c) 12 (d) 13



15 In the opposite figure :

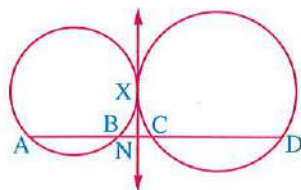
The two circles touch each other externally at X ,

\overleftrightarrow{AD} intersects one of the circles at A and B

and the other one at C and D

Let the common tangent to the two circles at X intersect \overleftrightarrow{AD} at N

Prove that : $\frac{NB}{NC} = \frac{ND}{NA}$



- 16 Two circles are intersecting at A and B , $C \in \overline{AB}$ and $C \notin \overline{AB}$, from C the two tangent segments \overline{CX} and \overline{CY} are drawn to touch the circles at X and Y respectively.

Prove that : $CX = CY$

17 In the opposite figure :

M and N are two circles touching externally at E

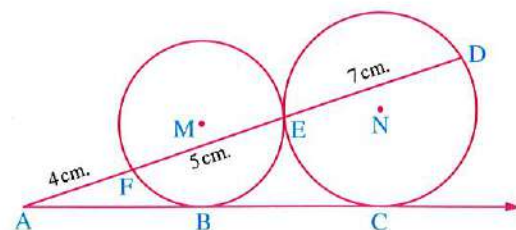
, \overline{AC} touches the circle M at B and touches

the circle N at C , \overline{AE} intersects the two

circles at F and D respectively ,

where AF = 4 cm. , FE = 5 cm. , ED = 7 cm.

Prove that : B is the midpoint of \overline{AC}



- 18 ABC is an acute-angled triangle , \overline{AD} , \overline{BE} are two intersecting heights at F

Prove that : $\frac{AE \times AC}{BF \times FE} = \frac{AD}{FD}$

- 19 A circle of centre O and its radius length equals 8 cm. , M is a point where $MO = 12$ cm. , from M a secant is drawn to intersect the circle at A and B where $A \in \overline{MB}$
If $AB = 11$ cm.

, find : (1) The length of \overline{MA}

(2) The length of the tangent segment to the circle from M « 5 cm. , $4\sqrt{5}$ cm. »

- 20 ABC is a triangle $D \in \overline{BC}$ where $BD = 5$ cm. and $DC = 4$ cm. If $AC = 6$ cm.

, prove that :

(1) \overline{AC} is a tangent segment to the circle passing through the points A , B and D

(2) $\Delta ACD \sim \Delta BCA$

(3) Area of (ΔABD) : area of $(\Delta ABC) = 5 : 9$

- 21 Two concentric circles at M , the lengths of their radii are 12 cm. and 7 cm.

\overline{AD} is a chord in the larger circle to intersect the smaller circle at B and C respectively.

Prove that : $AB \times BD = 95$

- 22 ABCD is a rectangle in which $AB = 6$ cm. and $BC = 8$ cm. , $\overline{BE} \perp \overline{AC}$ and intersects \overline{AC} at E and \overline{AD} at F

(1) **Prove that :** $(AB)^2 = AF \times AD$

(2) **Find the length of :** \overline{AF}

« 4.5 cm. »

- 23 \overline{AB} is a chord of length 8 cm. in a circle of centre M , $\overline{MC} \perp \overline{AB}$ to intersect it at C and intersect the circle at D. If $CD = 2$ cm. , calculate the length of the radius of the circle.

« 5 cm. »

- 24 \overline{AB} is a diameter in a circle , $C \in \overline{AB}$, $\overline{CX} \perp \overline{AB}$ to intersect the circle at X , \overline{DE} is a chord drawn in the circle passing through point C. **Prove that :** $(XC)^2 = DC \times CE$

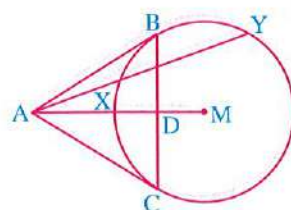
- 25 \overline{AB} is a diameter in a circle , \overline{CD} is a chord in it perpendicular to \overline{AB} to intersect it at N
The two chords \overline{AE} and \overline{AF} are drawn in two different sides from \overline{AB} to intersect \overline{CD} at X and Y respectively. **Prove that :** $AX \times AE = AY \times AF$

26 In the opposite figure :

A is a point outside the circle M, \overline{AB} and \overline{AC} are tangents to the circle, \overline{AY} intersects the circle at X and Y,

$$\overline{BC} \cap \overline{MA} = \{D\}$$

Prove that : $AX \times AY = AD \times AM$



27 \overline{AB} is a diameter in a circle, $C \in \overline{AB}$, C is located outside the circle where $BC = AB$, \overline{CD} is a tangent to the circle at D, \overline{AD} is drawn to intersect the tangent of the circle from point B at E

Prove that : $(CD)^2 = 2 AD \times AE$

28 ABC is a triangle, \overline{AD} bisects $\angle BAC$ and intersects \overline{BC} at D, $E \in \overline{AD}$ where $AD = DE$
If $(AD)^2 = DB \times DC$

, prove that : (1) $\triangle ECD \sim \triangle EAC$

$$(2) (EC)^2 = 2 (ED)^2$$



Problems that measure high standard levels of thinking

29 Choose the correct answer from those given :**(1) In the opposite figure :**

The radius of the circle (M)

, $ME = ED$, $EC = 3$ cm., $AE = 8$ cm.

, then $ME = \dots\dots\dots$ cm.

(a) 2 (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) $\frac{8}{3}$

(2) In the opposite figure :

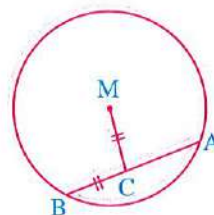
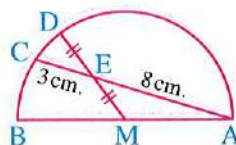
Circle M of diameter 12 cm.

, $MC = CB$, $AC = (BC + 1)$ cm.

, then $AB = \dots\dots\dots$ cm.

(a) 4 (b) 6

(c) 8 (d) 9

**(3) In the opposite figure :**

If \overline{AB} is a diameter in circle M

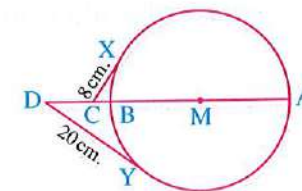
, \overline{CX} , \overline{DY} are two tangent segments of circle M

, $AB = 30$ cm., $CX = 8$ cm., $DY = 20$ cm.

, then $DC = \dots\dots\dots$ cm.

(a) 2 (b) 6

(c) 8 (d) 10

**(4) In the opposite figure :**

Two intersecting circles at C, E

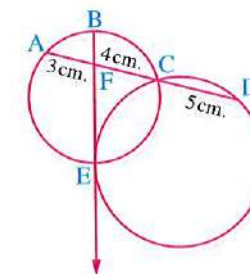
, \overline{BE} touches the larger circle at E

If $AF = 3$ cm., $FC = 4$ cm., $CD = 5$ cm.

, then $BE = \dots\dots\dots$ cm.

(a) 9 (b) 8

(c) 7 (d) 6

**(5) In the opposite figure :**

Two circles touching internally at B, \overline{AB} , \overline{AD}

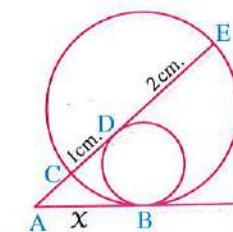
are two tangents to the smaller circle at B, D

If $CD = 1$ cm., $DE = 2$ cm., $AB = X$ cm.

, then $X = \dots\dots\dots$ cm.

(a) 2 (b) 3

(c) 2.5 (d) 3.5

**(6) In the opposite figure :**

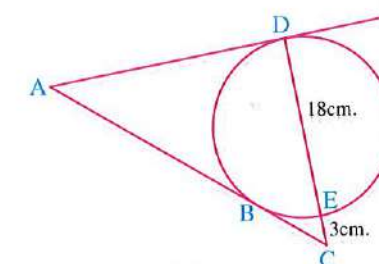
\overline{AD} , \overline{AB} are two tangents at D, B respectively

\overline{CE} intersects the circle at E, D

If $CE = 3$ cm., $ED = 18$ cm.

, then $(AC - AD) = \dots\dots\dots$ cm.

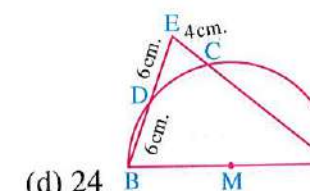
(a) 7 (b) $2\sqrt{7}$ (c) $3\sqrt{7}$

**(7) In the opposite figure :**

\overline{AB} is a diameter in a semicircle (M)

, then $r = \dots\dots\dots$ cm.

(a) 9 (b) 12 (c) 18



(8) In the opposite figure :

DC = cm.

- (a) 9 (b) 10
(c) 11 (d) 12

(9) In the opposite figure :

If $DY = 6$ cm. and $\frac{XE}{EY} = \frac{2}{3}$, then CX = cm.

- (a) 2 (b) 3
(c) 4 (d) 5

(10) In the opposite figure :

\overline{AB} is a diameter in circle (M), $E \in \overline{BA}$ to find the radius length of the circle it is sufficient to have

- (a) the perimeter of $\triangle EBC = 26$ cm. only.
(b) the perimeter of $\triangle EMC = 20$ cm. only.
(c) (a), (b) together.
(d) nothing of the previous.

(11) In the opposite figure :

The radius length of semicircle (M) = 10 cm.

, then ED = cm.

- (a) $\frac{50}{13}$ (b) $\frac{55}{13}$ (c) $\frac{57}{13}$ (d) $\frac{59}{13}$

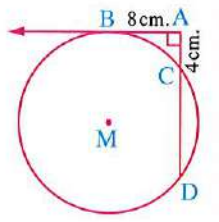
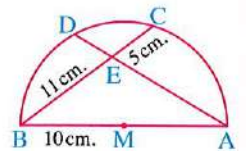
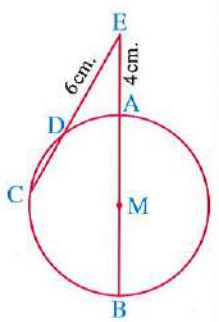
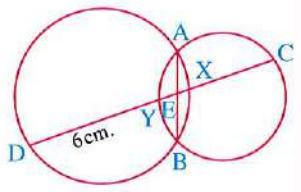
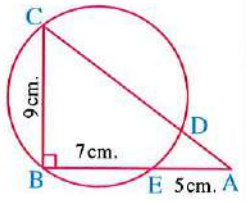
(12) In the opposite figure :

\overline{AB} is a tangent to the circle at B

, $AB = 8$ cm. , \overline{AC} is a secant to the circle M

at C and D , then the radius length of the circle (M) = cm.

- (a) 5 (b) 10 (c) 12 (d) 8



30 ABC is a triangle in which : $AB = 60$ mm. , $AC = 40$ mm. , $BC = 45$ mm. , take point $D \in \overline{AB}$ where $AD = 16$ mm. , $E \in \overline{AC}$ where $AE = 24$ mm.

(1) Prove that : $\triangle ADE \sim \triangle ACB$ and calculate the length of \overline{DE}

(2) If $\overline{DE} \cap \overline{BC} = \{N\}$, prove that : $\triangle DNB \sim \triangle CNE$ and calculate the length of each of : \overline{EN} , \overline{NC}

« 18 mm. , 21.6 mm. , 14.4 mm. »



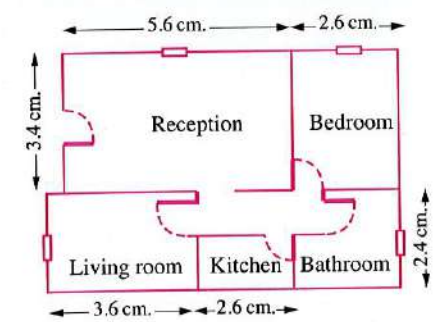
Life Applications on Unit Three



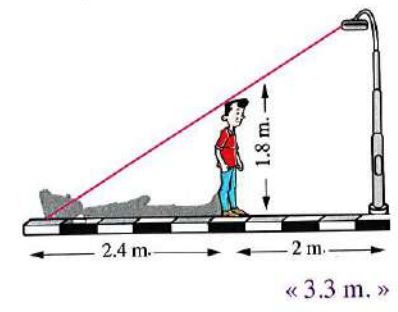
From the school book

1 The opposite figure shows the floor plan of a house with a drawing scale 1 : 150 Find :

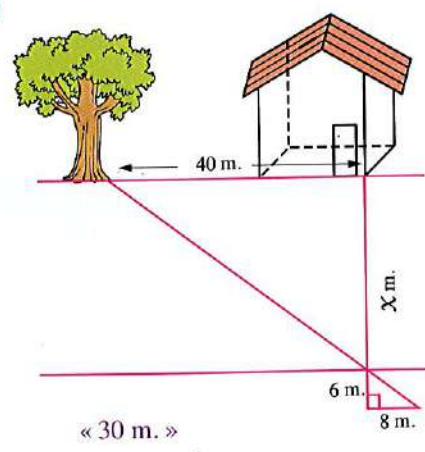
- (1) The dimensions of the reception.
(2) The dimensions of the bedroom.
(3) The area of the living room.
(4) The area of the house floor.



2 A man of height 1.8 m. stands against a light pole , at a distance 2 m. from its base. When the light is switched on , the length of the man's shadow is 2.4 m. Find the height of the pole.

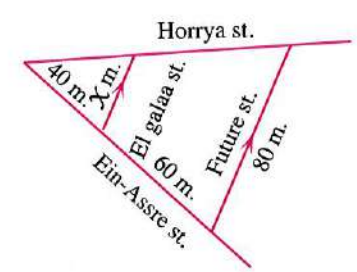


3 (1)



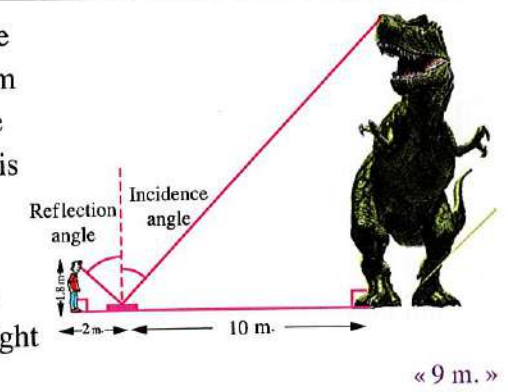
« 30 m. »

(2)



« 32 m. »

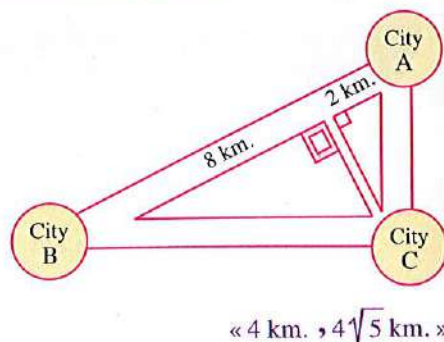
4 A man wanted to know the height of a dinosaur in one of the museums , he put a mirror 10 metres away from the foot of the dinosaur , then he moved back until he could see the head of the dinosaur in the mirror. At this moment he measured the distance from the mirror , it was 2 m. and the height of the man was 1.8 m. Given that the measure of the incidence angle equals the measure of the reflection angle , calculate the height of the dinosaur.



« 9 m. »

- 5 The opposite diagram shows the location of a gas station. It is required to be build on a highway at the intersection of a road that leads to city C and perpendicular to the highway between the two cities A and B , given that the highway between A and C is perpendicular to that between B and C

(1) How far is the gas station from city C ?
(2) What is the distance between B and C ?

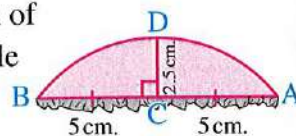


« 4 km. , $4\sqrt{5}$ km. »

- 6 The floor of a GYM rectangular hall of dimensions 8 m. and 12 m. was covered with wood , for 3200 pounds. Calculate (using similarity) the cost of covering a larger rectangular hall of dimensions 14 m. and 21 m. with the same kind of wood and price.

« L.E. 9800 »

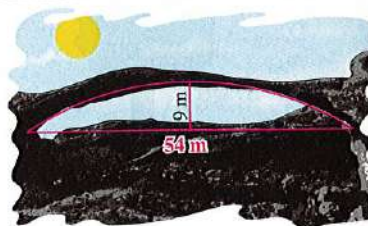
- 7 One of the architects found relics archaeological piece of wood is part of a circular wooden disc , this engineer wanted to know the length of the radius of the disc , so he appointed two points A , B on the circle , he found that $\overline{AB} = 10$ cm. , then from the point C which is the midpoint of \overline{AB} he draw $\overline{CD} \perp \overline{AB}$, he found that $\overline{CD} = 2.5$ cm. , so he could find the length of the radius geometrically. How he could so ?!



« 6.25 cm. »

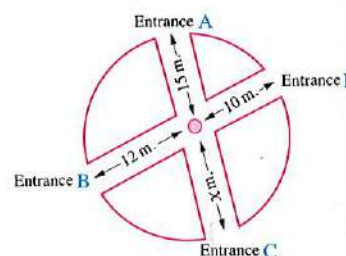
- 8 In one of the coastal areas , there is a ground layer in the form of a natural arc. The geologists found that , it is an arc of a circle , as in the opposite figure. Find the length of the radius of the circle arc.

« 45 m. »



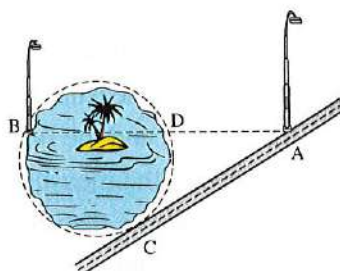
- 9 The opposite figure illustrates a plan of a circular garden involving two intersected roads at a fountain. How far is the fountain from the entrance C ?

« 8 m. »



- 10 In the opposite figure :

A road touches a circular lake , one of the engineers of the electricity company wants to put two light poles , one is on the road and the other lies in other side of the lake and joined between them by an electric wire. Show how to find the length of this wire.



Unit 4

The Triangle Proportionality Theorems

Unit Exercises

- Exercise 5 : Parallel lines and proportional parts.
Exercise 6 : Talis' theorem.
Exercise 7 : Angle bisector and proportional parts.
Exercise 8 : Follow : Angle bisector and proportional parts (Converse of theorem 3).
Exercise 9 : Applications of proportionality in the circle.

At the end of the unit :

- Life applications on unit four.

Parallel lines
and proportional parts

Test

yourself

From the school book

1 Using the opposite figure :

Choose the correct answer from those given :

(1) If $\frac{AD}{DB} = \frac{5}{3}$, then $\frac{AB}{BD} = \dots\dots\dots$

(a) $\frac{3}{5}$

(b) $\frac{8}{3}$

(c) $\frac{3}{8}$

(d) $\frac{5}{8}$

(2) If $\frac{AE}{AC} = \frac{4}{7}$, then $\frac{CE}{EA} = \dots\dots\dots$

(a) $\frac{7}{4}$

(b) $\frac{4}{3}$

(c) $\frac{2}{5}$

(d) $\frac{3}{4}$

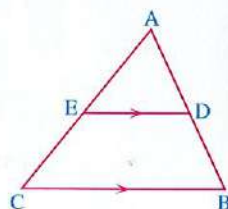
(3) If $\frac{DE}{BC} = \frac{3}{5}$, then $\frac{AD}{DB} = \dots\dots\dots$

(a) $\frac{5}{3}$

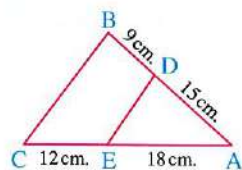
(b) 1.5

(c) $\frac{2}{3}$

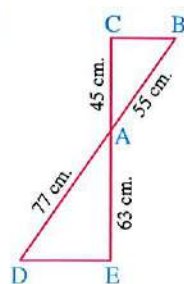
(d) $\frac{3}{4}$

2 In each of the following figures, is $\overline{DE} \parallel \overline{BC}$?

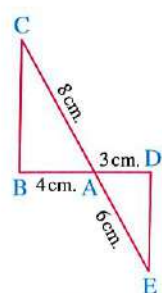
(1)



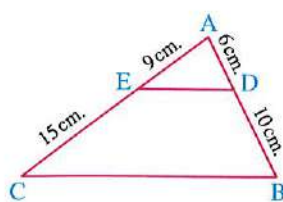
(2)



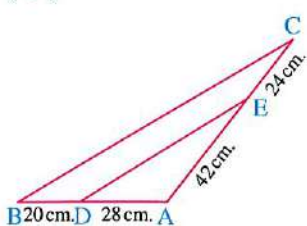
(3)



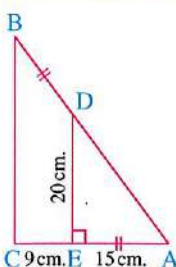
(4)



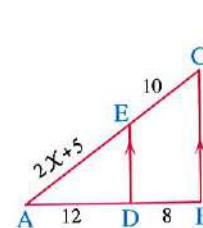
(5)



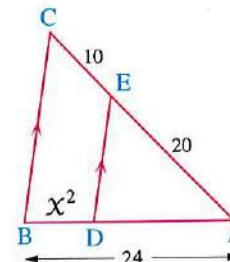
(6)

3 In each of the following figures, $\overline{DE} \parallel \overline{BC}$:Find the numerical value of x (Lengths are measured in centimetres) :

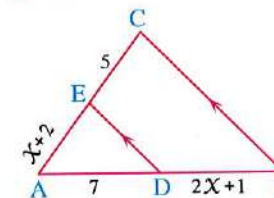
(1)



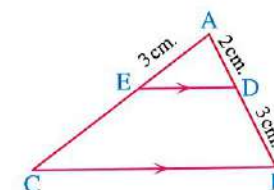
(2)



(3)

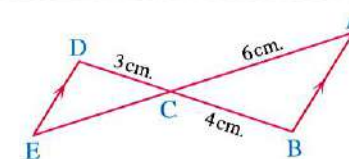


4 In the opposite figure :

If $\overline{DE} \parallel \overline{BC}$, $AD = 2$ cm.and $AE = DB = 3$ cm., find the length of : \overline{EC} 

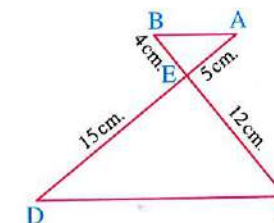
« 4.5 cm. »

5 In the opposite figure :

 $\overline{AB} \parallel \overline{DE}$, $\overline{AE} \cap \overline{BD} = \{C\}$, $AC = 6$ cm., $BC = 4$ cm. and $CD = 3$ cm.Find the length of : \overline{AE} 

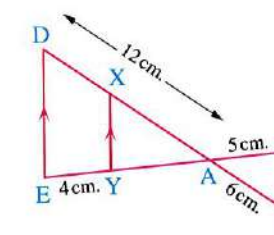
« 10.5 cm. »

6 In the opposite figure :

 $\overline{AD} \cap \overline{BC} = \{E\}$, $AE = 5$ cm., $BE = 4$ cm., $CE = 12$ cm. and $DE = 15$ cm.Prove that : $\overline{AB} \parallel \overline{CD}$ 7 $\overline{XY} \cap \overline{ZL} = \{M\}$, where $\overline{XZ} \parallel \overline{LY}$, if $XM = 9$ cm., $YM = 15$ cm. and $ZL = 36$ cm., find the length of : \overline{ZM}

« 13.5 cm. »

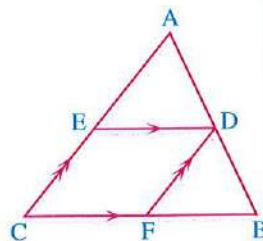
8 In the opposite figure :

 $\overline{CE} \cap \overline{BD} = \{A\}$, $X \in \overline{AD}$, $Y \in \overline{AE}$, where $\overline{XY} \parallel \overline{BC} \parallel \overline{ED}$, if $AB = 6$ cm., $AC = 5$ cm., $AD = 12$ cm. and $EY = 4$ cm., find the length of each of : \overline{AE} , \overline{DX} 

« 10 cm., 4.8 cm. »

- 9 For each of the following, use the opposite figure and the given data to find the value of x (Lengths are measured in centimetres) :

- (1) $AD = 4$, $BD = 8$, $CE = 6$ and $AE = x$
 (2) $AE = x$, $EC = 5$, $AD = x - 2$ and $DB = 3$
 (3) $AB = 21$, $BF = 8$, $FC = 6$ and $AD = x$
 (4) $AD = x$, $BF = x + 5$ and $2 DB = 3 FC = 12$



- 10 XYZ is a triangle in which $XY = 14$ cm., $XZ = 21$ cm., $L \in \overline{XY}$, where $XL = 5.6$ cm. and $M \in \overline{XZ}$ where $XM = 8.4$ cm. **Prove that : $\overline{LM} \parallel \overline{YZ}$**

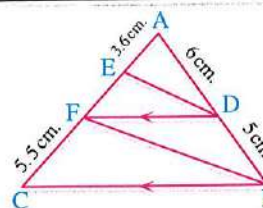
- 11 In the triangle ABC, $D \in \overline{AB}$, $E \in \overline{AC}$ and $5 AE = 4 EC$. If $AD = 10$ cm. and $DB = 8$ cm., is $\overline{DE} \parallel \overline{BC}$? Explain your answer.

- 12 ABCD is a trapezium in which $\overline{AD} \parallel \overline{BC}$, its diagonals \overline{AC} and \overline{BD} are intersected at M. If $AM = 2.5$ cm., $DB = 7\frac{1}{3}$ cm. and $MC = 3$ cm., find the length of each of : \overline{MD} and \overline{MB}

« $3\frac{1}{3}$ cm., 4 cm. »

- 13 In the opposite figure :

If $\overline{DF} \parallel \overline{BC}$, $AD = 6$ cm.,
 $BD = 5$ cm., $AE = 3.6$ cm. and $FC = 5.5$ cm.,
 then prove that : $\overline{DE} \parallel \overline{BF}$



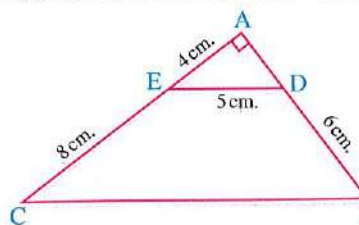
- 14 ABCD is a quadrilateral, its diagonals are intersected at E. If $AE = 6$ cm., $BE = 13$ cm., $EC = 10$ cm. and $ED = 7.8$ cm., prove that : ABCD is a trapezium.

- 15 In the opposite figure :

ABC is a right-angled triangle at A

- (1) Prove that : $\overline{DE} \parallel \overline{BC}$
 (2) Find the length of : \overline{BC}

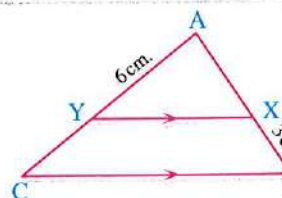
« 15 cm. »



- 16 In the opposite figure :

ABC is a triangle, in which $\overline{XY} \parallel \overline{BC}$

If $BX = 3$ cm., $AY = 6$ cm. and $\frac{AX + AY}{AB + AC} = \frac{3}{5}$,
 find the length of each of : \overline{AX} , \overline{CY}

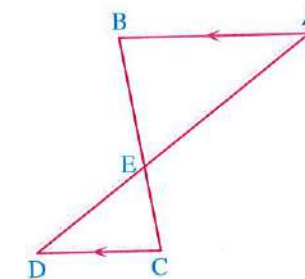


« 4.5 cm., 4 cm. »

- 17 Choose the correct answer from those given :

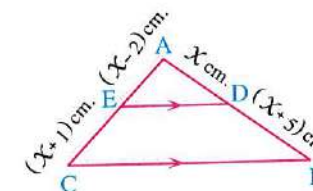
- (1) In the opposite figure :

If $\overline{AB} \parallel \overline{CD}$, $2 AE = 3 ED$
 $BE - CE = 4$ cm.,
 then $BC = \dots\dots\dots$ cm.
 (a) 18 (b) 20
 (c) 24 (d) 25



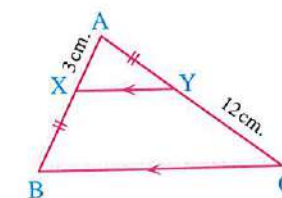
- (2) In the opposite figure :

$x = \dots\dots\dots$ cm.
 (a) 2 (b) 3
 (c) 4 (d) 5



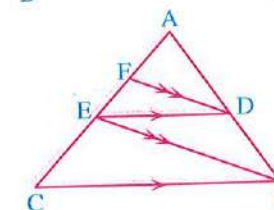
- (3) In the opposite figure :

$AC = \dots\dots\dots$ cm.
 (a) 15 (b) 16
 (c) 18 (d) 20



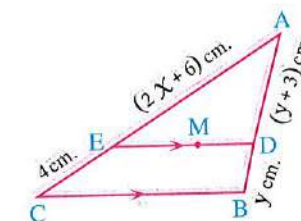
- (4) In the opposite figure :

If $\overline{DE} \parallel \overline{BC}$, $\overline{DF} \parallel \overline{BE}$
 then $AF \times AC = \dots\dots\dots$
 (a) AE (b) $(AE)^2$
 (c) $(DE)^2$ (d) $FE \times EC$



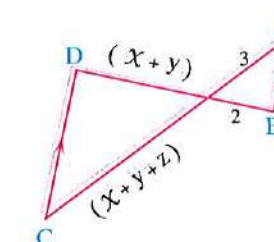
- (5) In the opposite figure :

If M is the point of intersection of medians of $\triangle ABC$
 then $2x + y = \dots\dots\dots$ cm.
 (a) 2 (b) 3
 (c) 4 (d) 5



- (6) In the opposite figure :

If $\overline{AB} \parallel \overline{CD}$, then $z = \dots\dots\dots$
 (a) $\frac{x-y}{2}$ (b) $\frac{x+y}{2}$
 (c) $5x + 5y$ (d) $\frac{x+y}{5}$



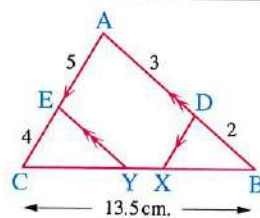
- 18 ABC is a triangle, $D \in \overline{AB}$, draw $\overline{DE} \parallel \overline{BC}$ to intersect \overline{AC} at E, then draw $\overline{EF} \parallel \overline{CD}$ to intersect \overline{AB} at F. **Prove that : $(AD)^2 = AF \times AB$**

- 19 ABCD is a quadrilateral, $E \in \overline{AC}$, draw $\overline{EF} \parallel \overline{CB}$ to intersect \overline{AB} at F, draw $\overline{EN} \parallel \overline{CD}$ to intersect \overline{AD} at N. **Prove that:** $\overline{FN} \parallel \overline{BD}$
- 20 **Prove that:** The line segment drawn between two midpoints of two sides in a triangle is parallel to the third side and its length is equal to a half of the length of this side.
- 21 ABCD is a parallelogram, $E \in \overline{BA}$, $E \notin \overline{AB}$, draw \overline{EC} to intersect \overline{AD} at F, \overline{BD} at M. **Prove that:** $(CM)^2 = MF \times ME$
- 22 ABCD is a parallelogram, $E \in \overline{CB}$, $E \notin \overline{CB}$, draw \overline{DE} to intersect \overline{AB} at N, then draw $\overline{BG} \parallel \overline{ED}$ to intersect \overline{CD} at G. **Prove that:** $\frac{AN}{NB} = \frac{CG}{GD}$
- 23 ABC is a triangle, $D \in \overline{AB}$, where $3AD = 2DB$ and $E \in \overline{AC}$, where $5CE = 3AC$ and \overline{AX} is drawn to intersect \overline{BC} at X, if $AF = 8$ cm. and $AX = 20$ cm. where $F \in \overline{AX}$. **Prove that:** The points D, F and E are collinear.
- 24 ABC is a triangle, $D \in \overline{BC}$, where $\frac{BD}{DC} = \frac{3}{4}$ and $E \in \overline{AD}$, where $\frac{AE}{AD} = \frac{3}{7}$, \overline{CE} is drawn to intersect \overline{AB} at X, $\overline{DY} \parallel \overline{CX}$ and intersects \overline{AB} at Y. **Prove that:** $AX = BY$

25 In the opposite figure :

ABC is a triangle in which : $\overline{DX} \parallel \overline{AC}$, $\overline{EY} \parallel \overline{AB}$,
 $BC = 13.5$ cm., $\frac{AD}{DB} = \frac{3}{2}$, $EC = \frac{4}{5} AE$

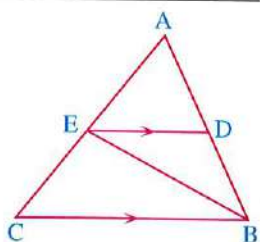
Find the length of: \overline{XY}



« 2.1 cm. »

- 26 ABC is a triangle, D is the midpoint of \overline{BC} , $M \in \overline{AD}$, draw $\overline{ME} \parallel \overline{AB}$ to intersect \overline{BC} at E, draw $\overline{MF} \parallel \overline{AC}$ to intersect \overline{BC} at F.

Prove that: D is the midpoint of \overline{EF} , if M is the point of intersection of the medians of $\triangle ABC$, then prove that : $EF = \frac{1}{3} BC$



27 In the opposite figure :

ABC is a triangle in which $\overline{DE} \parallel \overline{BC}$

Prove that: $\frac{\text{The area of } \triangle ADE}{\text{The area of } \triangle ABE} = \frac{\text{The area of } \triangle ABE}{\text{The area of } \triangle ABC}$



Problems that measure high standard levels of thinking

28 Choose the correct answer from those givens :

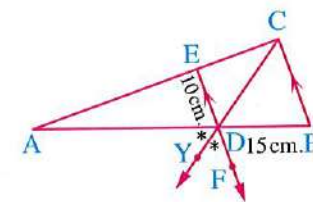
(1) In the opposite figure :

If $\overline{ED} \parallel \overline{BC}$, $m(\angle ADY) = m(\angle FDY)$

and $ED = 10$ cm., $BD = 15$ cm.

, then $AD = \dots\dots\dots$ cm.

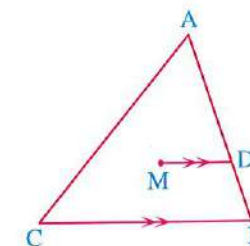
- (a) 20 (b) 25
(c) 30 (d) 45



(2) In the opposite figure :

If M is the point of intersection of medians of $\triangle ABC$,
 $\overline{DM} \parallel \overline{BC}$, then $\frac{DM}{BC} = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $\frac{2}{3}$ (d) $\frac{1}{4}$



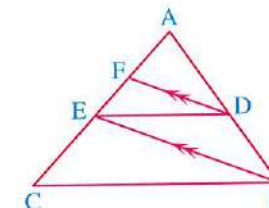
(3) In the opposite figure :

If $\overline{DF} \parallel \overline{BE}$, then to prove that

$\overline{DE} \parallel \overline{BC}$ it is sufficient

to get $\dots\dots\dots$

- (a) $\frac{AD}{DB} = \frac{3}{4}$ only (b) $AF \times AC = (AE)^2$ only
(c) (a), (b) together (d) Nothing of the previous



(4) In the opposite figure :

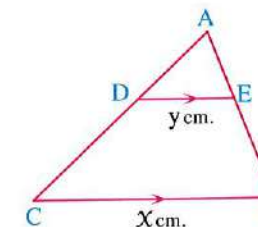
If $\overline{DE} \parallel \overline{BC}$, $DE = y$ cm.

, $BC = x$ cm., and $2x^2 - 3xy - 5y^2 = 0$

and $AB = 10$ cm., then

$EB = \dots\dots\dots$ cm.

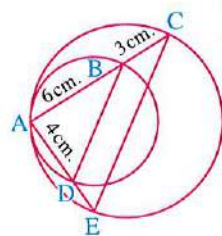
- (a) 3 (b) 4 (c) 6 (d) 8



(5) In the opposite figure :

Two circles touching internally at A
 , then ED = cm.

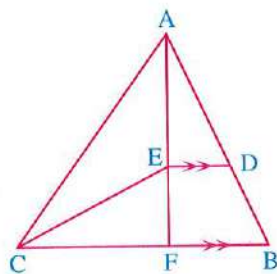
- (a) 2 (b) 3
 (c) 3.5 (d) 4



(6) In the opposite figure :

If the area of $(\triangle AEC) = 15 \text{ cm}^2$
 , the area of $(\triangle EFC) = 9 \text{ cm}^2$
 , AB = 16 cm. , then AD = cm.

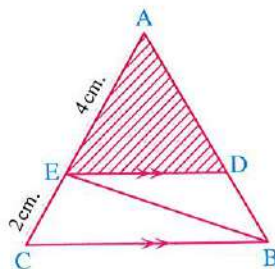
- (a) 6 (b) 10
 (c) 12 (d) 13



(7) In the opposite figure :

If $\overline{DE} \parallel \overline{BC}$ and the area
 of $(\triangle EBC) = 9 \text{ cm}^2$
 , then the area of $(\triangle ADE) = \dots \text{ cm}^2$

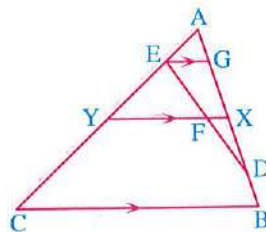
- (a) 6 (b) 12
 (c) 18 (d) 27



29 In the opposite figure :

ABC is a triangle , X is the midpoint of \overline{AB} ,
 Y is the midpoint of \overline{AC} , $D \in \overline{BX}$,
 $E \in \overline{AY}$, where $\frac{AD}{DB} = \frac{CE}{EA}$, $\overline{GE} \parallel \overline{XY} \parallel \overline{BC}$

Prove that : F is the midpoint of \overline{DE}



30 ABCD is a rectangle , its diagonals are intersected at M , E is the midpoint of \overline{AM} ,
 F is the midpoint of \overline{MC} , \overline{DE} is drawn to intersect \overline{AB} at X and \overline{DF} is drawn to intersect \overline{BC} at Y

Prove that : $\overline{XY} \parallel \overline{AC}$

Exercise

6

Talis' theorem



Test
yourself

From the school book

1 Write what each of the following ratios equals using the opposite figure :

$$(1) \frac{AB}{BC} = \frac{DE}{EF}$$

$$(2) \frac{AC}{BC} = \frac{EF}{DE}$$

$$(3) \frac{MA}{AB} = \frac{MD}{DE}$$

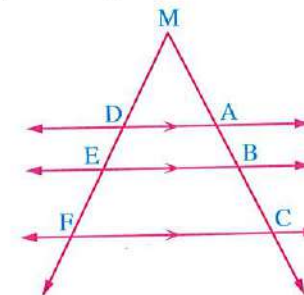
$$(4) \frac{AC}{AB} = \frac{DE}{EF}$$

$$(5) \frac{MB}{AB} = \frac{DE}{EF}$$

$$(6) \frac{MC}{AC} = \frac{MF}{FE}$$

$$(7) \frac{BC}{MB} = \frac{EF}{FE}$$

$$(8) \frac{DF}{MF} = \frac{AC}{CB}$$



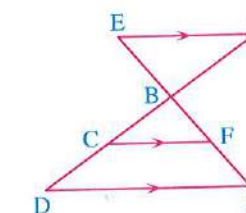
2 Choose the correct answer from those given :

(1) In the opposite figure :

AB : BC : CD
 = : :

- (a) AE : FC : MD
 (c) EB : BC : CD

- (b) EB : BF : FM
 (d) EB : EF : EM



(2) In the opposite figure :

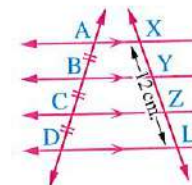
If AB = BC = CD ,

XL = 12 cm. , then XZ =

- (a) 4 cm. (b) YL

- (c) AC

- (d) BC

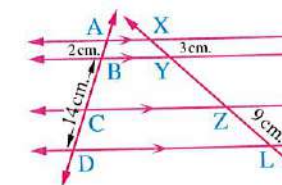


(3) In the opposite figure :

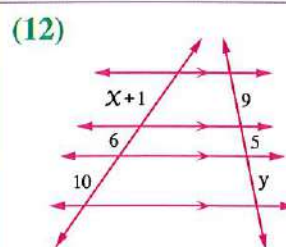
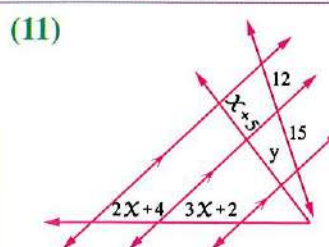
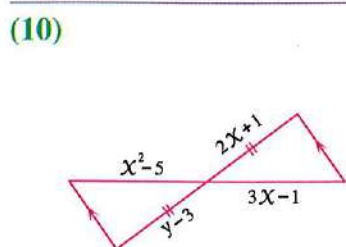
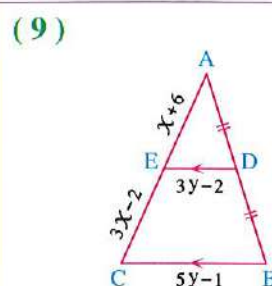
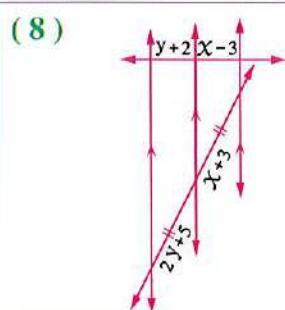
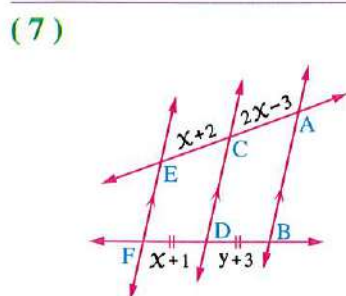
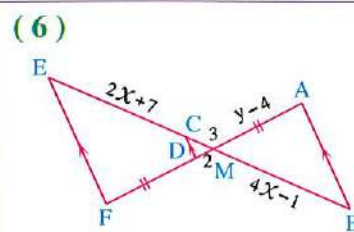
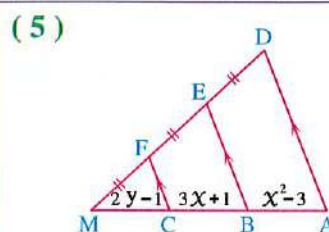
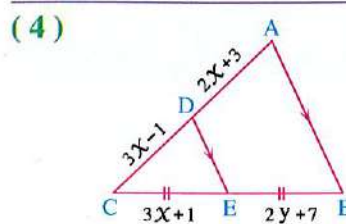
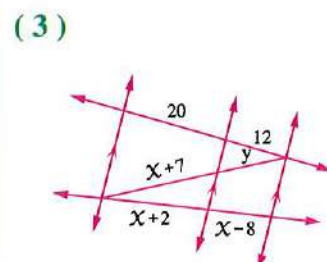
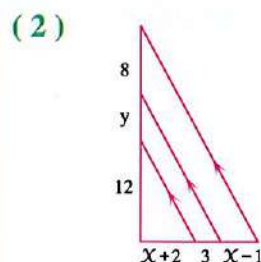
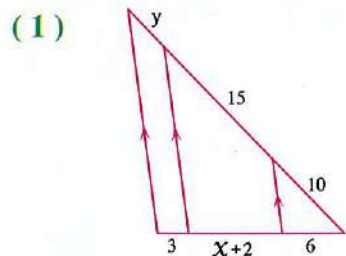
CD = cm.

- (a) 12
 (c) 14

- (b) 6
 (d) 5



- 3** In each of the following figures, calculate the numerical values of x and y
(Lengths are measured in centimetres):



- 4** In the opposite figure :

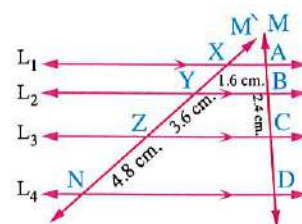
$$L_1 \parallel L_2 \parallel L_3 \parallel L_4,$$

M, \bar{M} are two transversals.

If $AB = 1.6 \text{ cm.}$, $BC = 2.4 \text{ cm.}$,

$YZ = 3.6 \text{ cm.}$, $ZN = 4.8 \text{ cm.}$

Calculate the length of each of : \bar{XY} and \bar{CD}



« 2.4 cm. , 3.2 cm. »

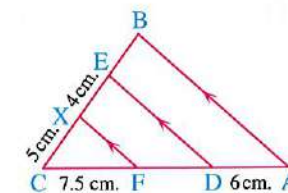
- 5** In the opposite figure :

If $\bar{AB} \parallel \bar{DE} \parallel \bar{FX}$,

$AD = 6 \text{ cm.}$, $EX = 4 \text{ cm.}$,

$FC = 7.5 \text{ cm.}$, $CX = 5 \text{ cm.}$

Find the length of each of : \bar{DF} , \bar{BE}



« 6 cm. , 4 cm. »

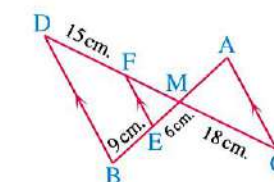
- 6** In the opposite figure :

$\bar{AB} \cap \bar{CD} = \{M\}$, $E \in \bar{MB}$,

$F \in \bar{MD}$ and $\bar{AC} \parallel \bar{FE} \parallel \bar{DB}$

Find : (1) The length of \bar{MF}

(2) The length of \bar{AM}



« 10 cm. , 10.8 cm. »

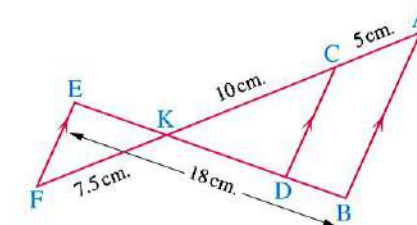
- 7** In the opposite figure :

If $\bar{AB} \parallel \bar{CD} \parallel \bar{EF}$,

$AC = 5 \text{ cm.}$, $CK = 10 \text{ cm.}$,

$KF = 7.5 \text{ cm.}$, $BE = 18 \text{ cm.}$

Find the length of each of : \bar{BD} , \bar{DK} and \bar{KE}



« 4 cm. , 8 cm. , 6 cm. »

- 8** $\bar{AB} \cap \bar{CD} = \{E\}$, $X \in \bar{AB}$, $Y \in \bar{CD}$, and $\bar{XY} \parallel \bar{BD} \parallel \bar{AC}$

Prove that : $AX \times ED = CY \times EB$

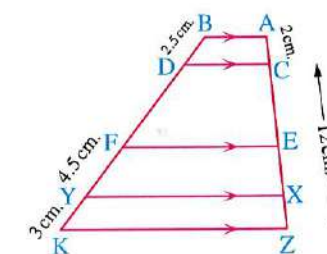
- 9** In the opposite figure :

$\bar{AB} \parallel \bar{CD} \parallel \bar{EF} \parallel \bar{XY} \parallel \bar{ZK}$,

$AC = 2 \text{ cm.}$, $BD = 2.5 \text{ cm.}$,

$FY = 4.5 \text{ cm.}$, $FK = 7.5 \text{ cm.}$, $CZ = 12 \text{ cm.}$

Find the length of each of : \bar{EX} , \bar{XZ} , \bar{CE} and \bar{DF}



« 3.6 cm. , 2.4 cm. , 6 cm. , 7.5 cm. »

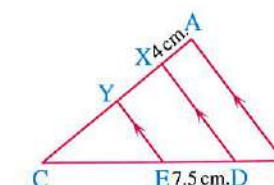
- 10** In the opposite figure :

$\bar{AB} \parallel \bar{DX} \parallel \bar{EY}$,

$AX : XY : YC = 2 : 3 : 5$

If $DE = 7.5 \text{ cm.}$, $AX = 4 \text{ cm.}$

, find the length of each of : \bar{BD} , \bar{CE} and \bar{AC}



« 5 cm. , 12.5 cm. , 20 cm. »

- 11** ABC is a triangle, $D, E \in \overline{AB}$, let $\overline{DX}, \overline{EY}$ be drawn parallel to \overline{BC} and intersect \overline{AC} at X and Y respectively, if $AD = \frac{1}{2} BE$, $DE = 3 AD$, $AC = 24$ cm.

Find the length of each of : \overline{AX} , \overline{XY} and \overline{YC}

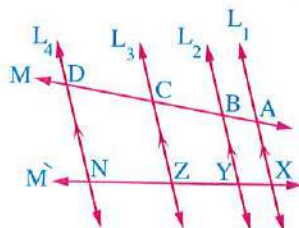
« 4 cm. , 12 cm. , 8 cm. »

- 12** In the opposite figure :

$L_1 \parallel L_2 \parallel L_3 \parallel L_4$ and M, \hat{M} are two transversals.

If $\frac{AB}{BC} = \frac{1}{2}$, $BC = \frac{4}{5} CD$ and $XN = 16.5$ cm.

Find the length of each of : \overline{XY} , \overline{YZ} and \overline{ZN}



« 3 cm. , 6 cm. , 7.5 cm. »

- 13** ABC is a triangle, $D \in \overline{AB}$ where $\frac{AD}{DB} = \frac{3}{5}$, let $E \in \overline{BA}$ outside the triangle such that : $AE = \frac{1}{2} AB$, let $\overline{DX}, \overline{EY}$ be drawn parallel to \overline{BC} to intersect \overline{AC} at X, Y respectively.

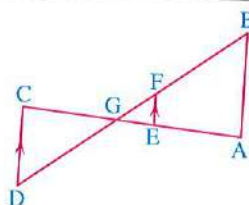
If $AY = 14$ cm. Find the length of each of : \overline{AX} , \overline{AC}

« 10.5 cm. , 28 cm. »

- 14** In the opposite figure :

$\overline{EF} \parallel \overline{CD}$, $\frac{AG}{GC} = \frac{DG}{GF}$

Prove that : $(GC)^2 = GA \times GE$



- 15** ABCD is a trapezium in which $\overline{AB} \parallel \overline{DC}$ and M is the midpoint of \overline{AD} , draw a straight line passing through the point M and parallel to \overline{DC} to intersect the diagonal \overline{BD} at N, diagonal \overline{AC} at E and the side \overline{BC} at F

(1) Show that the points N, E, F are the midpoints of \overline{BD} , \overline{AC} and \overline{BC} respectively.

(2) Prove that : $MF = \frac{1}{2} (AB + DC)$

- 16** ABCD is a quadrilateral in which $\overline{AB} \parallel \overline{CD}$, its diagonals intersect at M and E is the midpoint of \overline{BC} , $\overline{EF} \parallel \overline{BA}$ and intersects \overline{BD} at X, \overline{AC} at Y and \overline{AD} at F

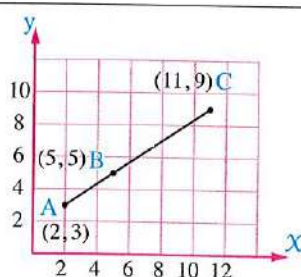
Prove that : (1) $EY = \frac{1}{2} AB$

(2) $\frac{AY}{CM} = \frac{BX}{DM}$

- 17** Logical thinking :

From the figure, find the value of $\frac{AB}{BC}$ in different methods, if possible.

Did you get the same result ?



Problems that measure high standard levels of thinking

- 18** Choose the correct answer from those given :

(1) In the opposite figure :

If $x^2 + y^2 = 57$

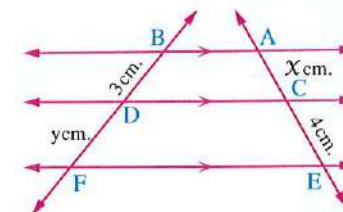
, then $x + y = \dots\dots\dots$ cm.

(a) 7

(b) 9

(c) 11

(d) 12



(2) In the opposite figure :

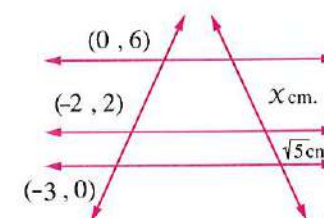
$x = \dots\dots\dots$ cm.

(a) $\sqrt{5}$

(b) $2\sqrt{5}$

(c) $3\sqrt{5}$

(d) $4\sqrt{5}$



(3) In the opposite figure :

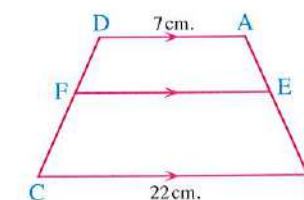
If $\frac{AE}{EB} = \frac{2}{3}$, then $EF = \dots\dots\dots$ cm.

(a) 9

(b) 11

(c) 13

(d) 15



(4) In the opposite figure :

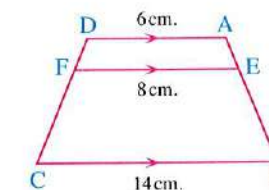
$\frac{AE}{EB} = \dots\dots\dots$

(a) $\frac{3}{4}$

(b) $\frac{4}{7}$

(c) $\frac{3}{7}$

(d) $\frac{1}{3}$



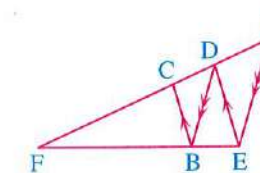
- 19** ABC is a triangle, M is the midpoint of \overline{BC} , let $K \in \overline{AM}$, draw $\overline{KE} \parallel \overline{AB}$ to intersect \overline{BC} at E, draw $\overline{KG} \parallel \overline{AC}$ to intersect \overline{BC} at G

Prove that : M is the midpoint of \overline{EG} , if K is the point of intersection of the medians of $\triangle ABC$, then prove that : $BE = EG = GC = \frac{1}{3} BC$

- 20** In the opposite figure :

$\overline{ED} \parallel \overline{BC}$, $\overline{DB} \parallel \overline{EX}$

Prove that : $\left(\frac{FB}{FE}\right)^2 = \frac{FC}{FX}$



- 21** ABCD is a parallelogram, draw \overline{DE} to intersect \overline{AC} , \overline{AB} at X, E respectively, draw \overline{DF} to intersect \overline{AC} , \overline{BC} at Y, F respectively. If $AX = CY$, prove that : $\overline{EF} \parallel \overline{XY}$

Angle bisector
and proportional partsTest
yourself

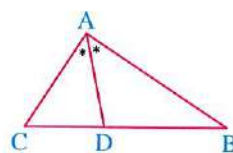
From the school book

1 Choose the correct answer from those given :

(1) In the opposite figure :

 \overrightarrow{AD} bisects $\angle A$, then $AB \times CD = \dots\dots\dots$

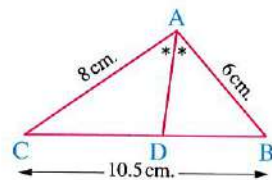
- (a) $AC \times BD$ (b) $(AD)^2$
(c) $AD \times BD$ (d) $AC \times AB$



(2) In the opposite figure :

 $BD = \dots\dots\dots$ cm.

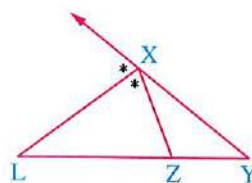
- (a) 4 (b) $\frac{2}{3}$
(c) 4.5 (d) 45



(3) In the opposite figure :

 \overrightarrow{XL} bisects the exterior angle X, then $\frac{YL}{YX} = \dots\dots\dots$

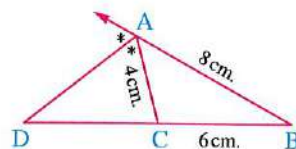
- (a) $\frac{YZ}{ZL}$ (b) $\frac{YL}{LZ}$
(c) $\frac{LZ}{ZX}$ (d) $\frac{XZ}{XY}$



(4) In the opposite figure :

 $CD = \dots\dots\dots$ cm.

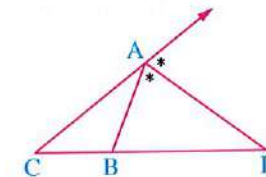
- (a) 2 (b) 6
(c) 4 (d) 8



(5) In the opposite figure :

If $AB : AC = 2 : 3$, then $BD : BC = \dots\dots\dots$

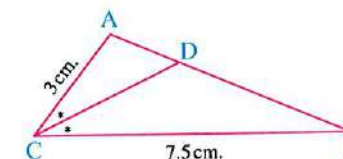
- (a) 2 : 1 (b) $\frac{3}{2}$
(c) $\frac{2}{3}$ (d) $\frac{1}{2}$



(6) In the opposite figure :

 \overrightarrow{CD} bisects $\angle C$, $AC = 3$ cm., $BC = 7.5$ cm., then $AD : BD = \dots\dots\dots$

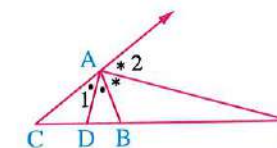
- (a) $\frac{3}{5}$ (b) $\frac{2}{3}$
(c) $\frac{2}{5}$ (d) $\frac{5}{2}$



(7) In the opposite figure :

If $m(\angle 1) = 36^\circ$, then $m(\angle 2) = \dots\dots\dots^\circ$

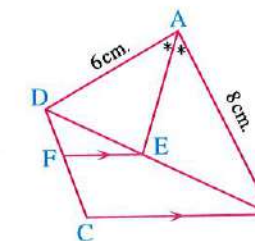
- (a) 36 (b) 40
(c) 54 (d) 108



(8) In the opposite figure :

 $\frac{DF}{FC} = \frac{\dots\dots\dots}{\dots\dots\dots}$

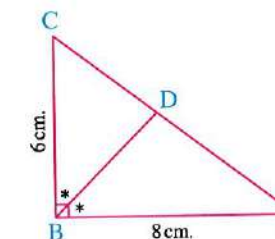
- (a) $\frac{4}{3}$ (b) $\frac{8}{7}$
(c) $\frac{2}{3}$ (d) $\frac{3}{4}$



(9) In the opposite figure :

 $AD = \dots\dots\dots$ cm.

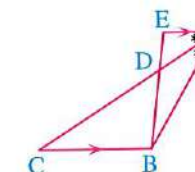
- (a) $5\frac{5}{7}$ (b) $6\frac{3}{4}$
(c) 5 (d) $\frac{4}{3}$



(10) In the opposite figure :

If $AC = 3 AD$, then $AB : AE = \dots\dots\dots : \dots\dots\dots$

- (a) 3 : 1 (b) 1 : 2
(c) 4 : 3 (d) 2 : 1



(11) In the opposite figure :

AC = cm.

- (a) 3 (b) 4
(c) 6 (d) 8

(12) In the opposite figure :

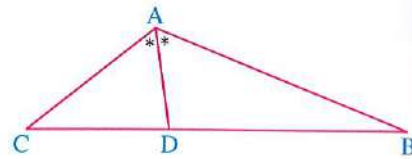
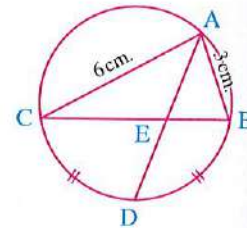
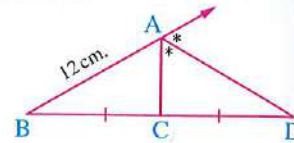
$\frac{BE}{BC} = \dots\dots\dots$

- (a) $\frac{1}{2}$ (b) 2
(c) $\frac{1}{3}$ (d) 3

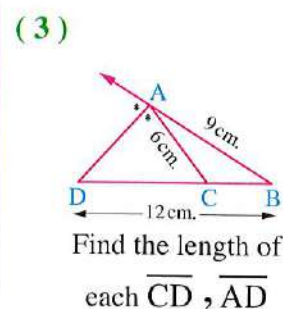
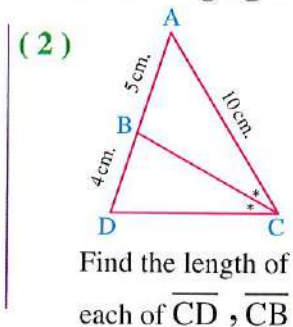
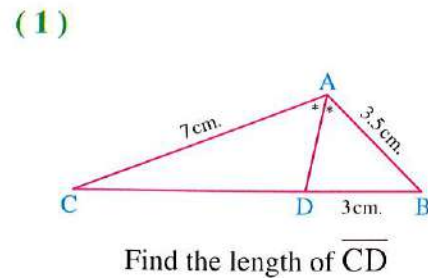
(13) In the opposite figure :

If $AB : AC : BC = 5 : 3 : 7$, then $BD : DC = \dots\dots\dots$

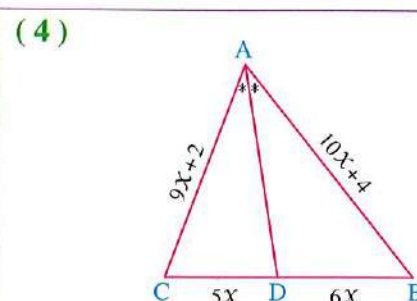
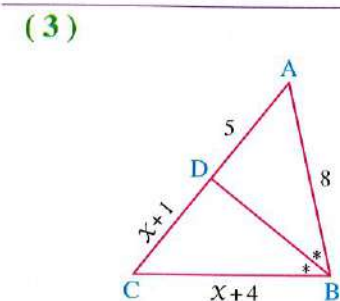
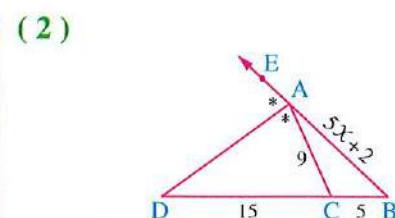
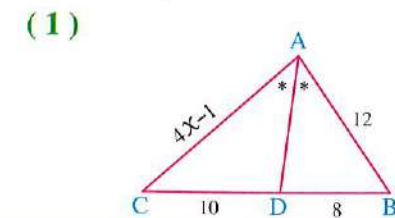
- (a) $\frac{5}{3}$ (b) $\frac{5}{7}$
(c) $\frac{3}{5}$ (d) $\frac{3}{7}$



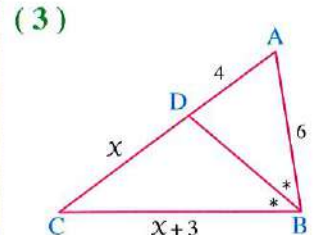
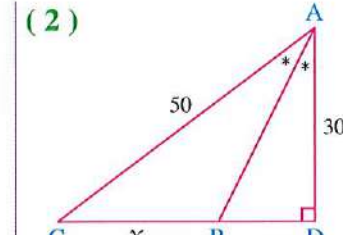
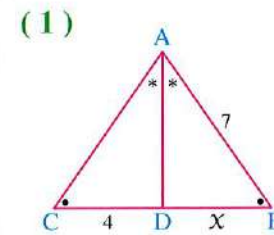
2 Find the requirement under each of the following figures :



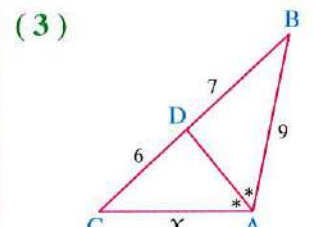
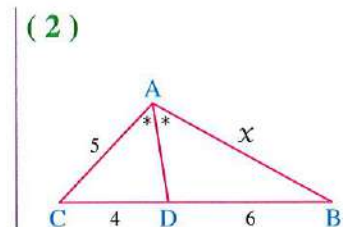
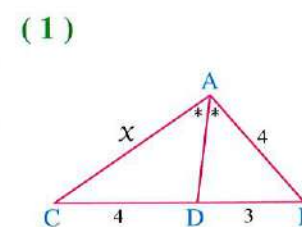
3 In each of the following figures, find the value of x (Lengths are measured in centimetres) :



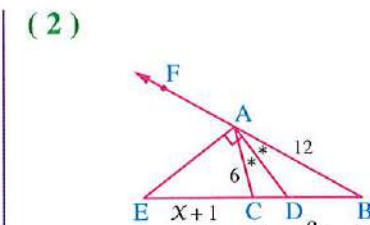
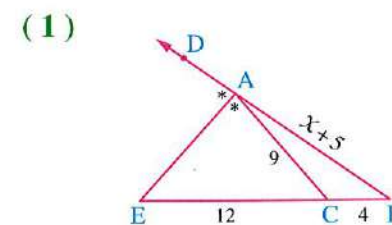
4 In each of the following figures, find the value of x (Lengths are measured in centimetres), then find the perimeter of $\triangle ABC$:



5 In each of the following figures, calculate the value of x and the length of \overline{AD} (Lengths are measured in centimetres) :



6 In each of the following two figures, calculate the value of x and the length of \overline{AE} (Lengths are measured in centimetres) :



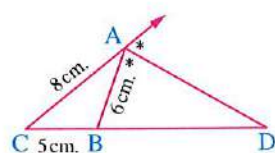
7 ABC is a triangle in which : $AB = 4$ cm. , $BC = 6$ cm. , draw \overline{BD} bisects $\angle ABC$ and intersects \overline{AC} at D , if $AD = 2.4$ cm. , find the length of : \overline{AC} « 6 cm. »

8 ABC is a triangle in which : $AB = 8$ cm. , $AC = 6$ cm. , $BC = 7$ cm. , \overline{AD} bisects $\angle BAC$ and intersects \overline{BC} at D Find the length of each of : \overline{DB} , \overline{DC} « 4 cm. , 3 cm. »

9 In the opposite figure :

ABC is a triangle in which \overline{AD} bisects the exterior angle at A and intersects \overline{CB} at D , if $AB = 6$ cm. , $AC = 8$ cm. , $BC = 5$ cm.

Find the length of each of : \overline{BD} , \overline{AD}



« 15 cm. , $6\sqrt{7}$ cm. »

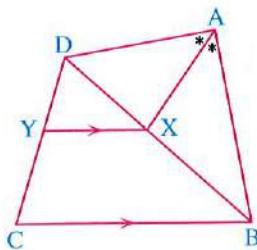
- 10 ABC is a triangle in which $AB = 3$ cm, $BC = 4$ cm, $CA = 6$ cm, \overrightarrow{AD} bisects the exterior angle at A and intersects \overrightarrow{BC} at D, find the length of each of : \overline{CD} , \overline{AD} « 8 cm, $\sqrt{14}$ cm. »

- 11 ABC is a triangle, its perimeter is 27 cm, \overrightarrow{BD} bisects $\angle B$ and intersects \overline{AC} at D. If $AD = 4$ cm, and $CD = 5$ cm, find the length of each of : \overline{AB} , \overline{BC} and \overline{BD}
« 8 cm, 10 cm, $2\sqrt{15}$ cm. »

12 In the opposite figure :

ABCD is a quadrilateral, draw \overrightarrow{AX} bisects $\angle A$ and intersects \overline{BD} at X, then draw $\overline{XY} \parallel \overline{BC}$ and intersects \overline{CD} at Y

Prove that : $\frac{DY}{YC} = \frac{AD}{AB}$

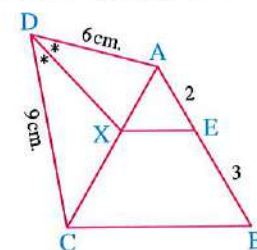


13 In the opposite figure :

ABCD is a quadrilateral in which \overrightarrow{DX} bisects $\angle D$,

$AE : EB = 2 : 3$, $AD = 6$ cm, $DC = 9$ cm,

, prove that : $\overline{EX} \parallel \overline{BC}$



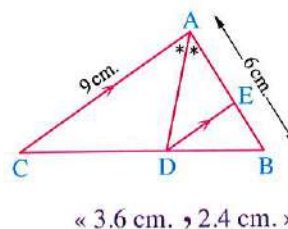
14 In the opposite figure :

\overrightarrow{AD} bisects $\angle BAC$, $\overline{ED} \parallel \overline{AC}$

Prove that : $\frac{BE}{EA} = \frac{BA}{AC}$

and if $AC = 9$ cm, $AB = 6$ cm,

, find the length of each of : \overline{AE} and \overline{BE}

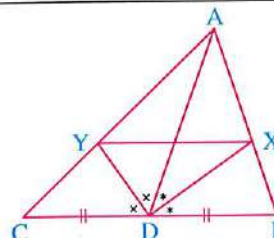


15 In the opposite figure :

\overline{AD} is a median of $\triangle ABC$,

\overrightarrow{DX} bisects $\angle ADB$, \overrightarrow{DY} bisects $\angle ADC$

Prove that : $\overline{XY} \parallel \overline{BC}$



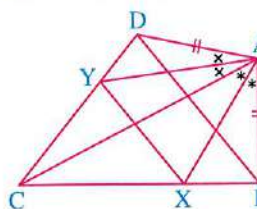
16 In the opposite figure :

ABCD is a quadrilateral in which $AB = AD$,

\overrightarrow{AX} bisects $\angle BAC$ and intersects \overline{BC} at X,

\overrightarrow{AY} bisects $\angle DAC$ and intersects \overline{CD} at Y

Prove that : $\overline{XY} \parallel \overline{BD}$



- 17 ABC is a right-angled triangle at B, draw \overrightarrow{AD} bisects $\angle A$, and intersects \overline{BC} at D. If the length of \overline{BD} equals 24 cm, $BA : AC = 3 : 5$, find the perimeter of $\triangle ABC$ « 192 cm. »

- 18 ABC is a triangle in which $AB = 8$ cm, $AC = 4$ cm, and $BC = 6$ cm, \overrightarrow{AD} bisects $\angle A$ and intersects \overline{BC} at D, \overrightarrow{AE} bisects the exterior angle at A and intersects \overline{BC} at E. Find the length of each of : \overline{DE} , \overline{AD} and \overline{AE} « 8 cm, $2\sqrt{6}$ cm, $2\sqrt{10}$ cm. »

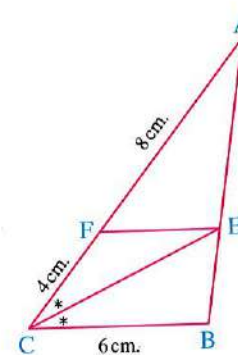
- 19 ABC is a triangle in which $AB = 3$ cm, $BC = 7$ cm, $CA = 6$ cm, \overrightarrow{AD} bisects $\angle A$ and intersects \overline{BC} at D, \overrightarrow{AE} bisects the exterior angle of the triangle at A and intersects \overline{CB} at E

(1) Prove that : \overline{AB} is a median in the triangle ACE

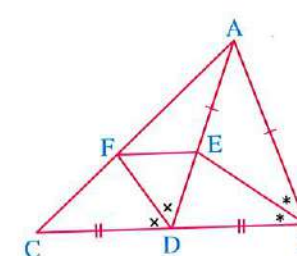
(2) Find the ratio of : The area of $\triangle ADE$ to the area of $\triangle ACE$ « $\frac{2}{3}$ »

- 20 In each of the following two figures, prove that $\overline{EF} \parallel \overline{BC}$:

(1)



(2)



- 21 ABC is a triangle in which : $AB > AC$, $D \in \overline{AB}$, where $BD = AC$, draw \overrightarrow{AE} bisects $\angle BAC$ and intersects \overline{DC} at E, then draw $\overrightarrow{EF} \parallel \overline{BA}$ and intersects \overline{AC} at F

Prove that : $\overline{DF} \parallel \overline{BC}$

- 22 ABCD is a parallelogram, $X \in \overline{AD}$, \overrightarrow{CX} is drawn to intersect \overline{BA} at Y and $\angle DCX$ is bisected by \overrightarrow{CZ} which intersected \overline{AD} at Z. Prove that : $\frac{AY}{YX} = \frac{DZ}{ZX}$

- 23 ABC is a triangle, \overrightarrow{AD} bisects $\angle BAC$ and intersects \overline{BC} at D, the two bisectors \overrightarrow{AE} , \overrightarrow{AF} bisect the two angles $\angle BAD$, $\angle CAD$ respectively and intersect \overline{BC} at E and F respectively. Prove that : $\frac{BE}{ED} \times \frac{DF}{FC} = \frac{BD}{DC}$

- 24 ABC is a triangle, draw \overrightarrow{AD} , \overrightarrow{BE} , \overrightarrow{CF} to bisect $\angle A$, $\angle B$ and $\angle C$ and to intersect \overline{BC} , \overline{AC} and \overline{AB} at D, E and F respectively. Prove that : $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$

25 Choose the correct answer from those given :

(1) In the opposite figure :

$AB = \dots\dots\dots$ cm.

- (a) 4 (b) 5
(c) 6 (d) 7

(2) In the opposite figure :

$\frac{AF}{FC} = \dots\dots\dots$

- (a) $\frac{2}{3}$ (b) $\frac{3}{4}$
(c) $\frac{4}{5}$ (d) $\frac{1}{2}$

(3) In the opposite figure :

If $AC - AB = 6$ cm. , then $AC = \dots\dots\dots$ cm.

- (a) 13 (b) 14
(c) 15 (d) 16

(4) In the opposite figure :

$AD = \dots\dots\dots$ cm.

- (a) 10 (b) $4\sqrt{5}$
(c) $6\sqrt{5}$ (d) $9\sqrt{2}$

(5) In the opposite figure :

$XC = \dots\dots\dots$ cm.

- (a) 1 (b) 2
(c) 3 (d) 4

(6) In the opposite figure :

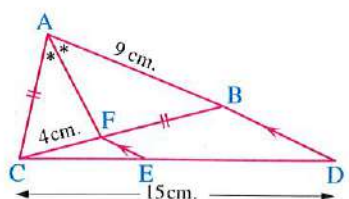
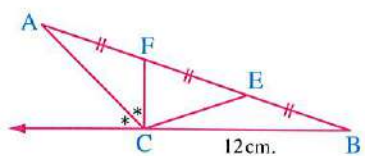
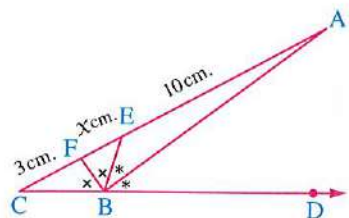
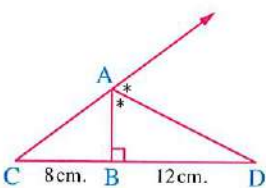
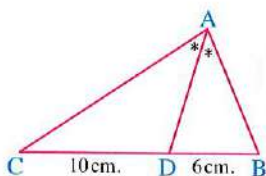
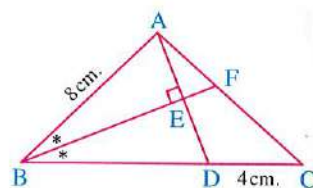
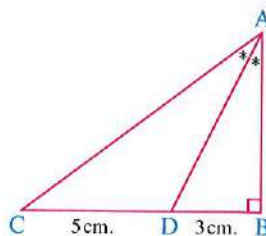
$CF = \dots\dots\dots$ cm.

- (a) 3 (b) 4
(c) 5 (d) 6

(7) In the opposite figure :

$ED = \dots\dots\dots$ cm.

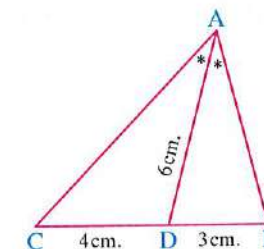
- (a) 6 (b) 8
(c) 9 (d) 12



(8) In the opposite figure :

$AC = \dots\dots\dots$ cm.

- (a) 12 (b) 10
(c) 9 (d) 8



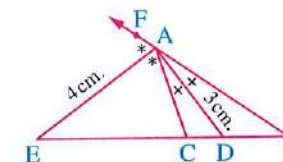
(9) In the opposite figure :

\overrightarrow{AD} bisects $\angle A$ internally , \overrightarrow{AE} bisects $\angle A$ externally ,

$AD = 3$ cm. , $AE = 4$ cm.

, then $DE = \dots\dots\dots$ cm.

- (a) 3 (b) 4
(c) 5 (d) 6



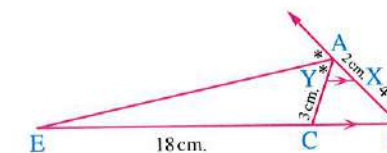
26 In the opposite figure : $\overline{XY} \parallel \overline{BC}$, $AX = 2$ cm. ,

$XB = 4$ cm. , $YC = 3$ cm. Find the length of : \overline{AY}

If \overline{AE} bisects the exterior angle of the triangle at A

and intersects \overline{BC} at E , where $CE = 18$ cm. ,

find the length of : \overline{BC}



« 1.5 cm. , 6 cm. »

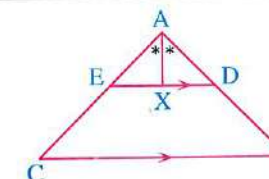
27 ABCD is a quadrilateral in which $AB = BD$, $AD = DC$, \overline{AE} bisects $\angle BAD$ and intersects \overline{BD} at E , \overline{DF} bisects $\angle BDC$ and intersects \overline{BC} at F

Prove that : $\overline{EF} \parallel \overline{DC}$

28 In the opposite figure : $\overline{DE} \parallel \overline{BC}$, \overline{AX} bisects $\angle DAE$

Prove that : (1) $\frac{DX}{XE} = \frac{DB}{EC}$

(2) $\frac{\text{The area of } \triangle ADX}{\text{The area of } \triangle AEX} = \frac{AB}{AC}$



29 ABCD is a parallelogram , its diagonals intersect at M , draw \overline{AX} to bisect $\angle BAD$ and to intersect \overline{BD} at X , draw \overline{DY} to bisect $\angle ADC$ and to intersect \overline{AC} at Y

Prove that : $\overline{XY} \parallel \overline{AD}$

30 \overline{AB} is a chord in a circle , let D \in the major arc \widehat{AB} such that $\frac{AD}{DB} = \frac{2}{3}$ and let E be the midpoint of the minor arc \widehat{AB} , draw \overline{DE} to intersect \overline{AB} at C , find the ratio between the area of $\triangle ADE$ and the area of $\triangle BDE$

« $\frac{2}{3}$ »

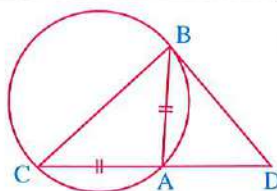
- 31 \overline{AB} is a diameter of a circle M , $C \in$ this circle, draw a tangent to the circle M at C to intersect \overline{AB} at E and to intersect the tangent to the circle M from A at D

Prove that : $\frac{AM}{ME} = \frac{DC}{DE}$

- 32 In the opposite figure :

$AB = AC$, \overline{BD} is a tangent to the circle at B

Prove that : $DB \times BA = DA \times BC$



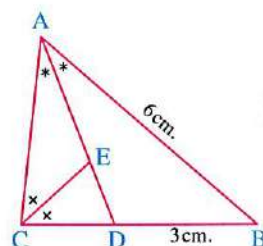
Problems that measure high standard levels of thinking

- 33 Choose the correct answer from those given :

- (1) In the opposite figure :

$$\frac{AE}{ED} = \dots\dots\dots$$

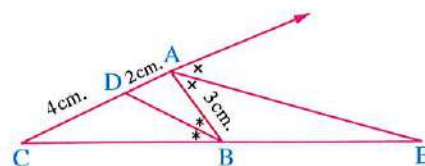
- (a) $\frac{1}{2}$ (b) 2
(c) 3 (d) $\frac{2}{3}$



- (2) In the opposite figure :

$BE = \dots\dots\dots$ cm.

- (a) 6 (b) 8
(c) 9 (d) 10



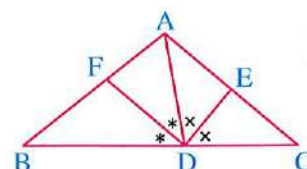
- (3) In the opposite figure :

If $3AE = 4EC$

, $2AF = 3FB$

, $BC = 17$ cm. , then $CD = \dots\dots\dots$ cm.

- (a) 7 (b) 8
(c) 9 (d) 10

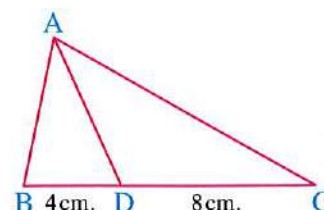


- (4) In the opposite figure :

If $m(\angle B) = 2m(\angle DAB) = 2m(\angle DAC)$

, then $AB = \dots\dots\dots$ cm.

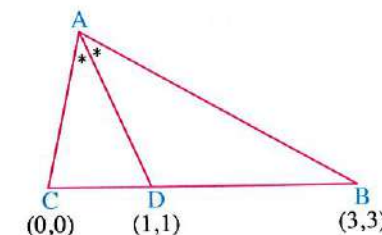
- (a) 4 (b) 6
(c) 8 (d) 9



- (5) In the opposite figure :

$$\frac{AC}{AB} = \dots\dots\dots$$

- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $\frac{1}{4}$ (d) $\frac{2}{3}$

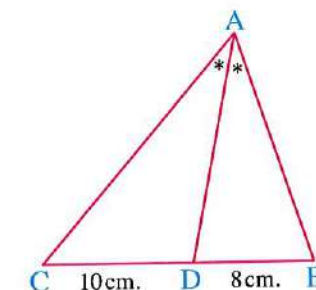


- (6) In the opposite figure :

If \overline{AD} bisects $\angle BAC$

which of the following conditions sufficient to find the length of \overline{AB} ?

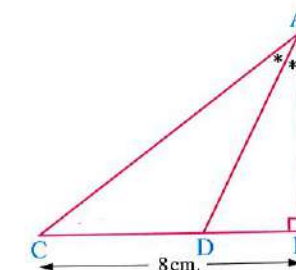
- (a) $AC - AB = 5$ cm.
(b) The perimeter of $\triangle ABC = 54$ cm.
(c) $AD = 4\sqrt{15}$ cm.
(d) Anything of the previous.



- (7) In the opposite figure :

If $\frac{\text{The area of } (\triangle ABD)}{\text{The area of } (\triangle ADC)} = \frac{3}{5}$
, then $AB = \dots\dots\dots$ cm.

- (a) 5 (b) 6
(c) 8 (d) 10

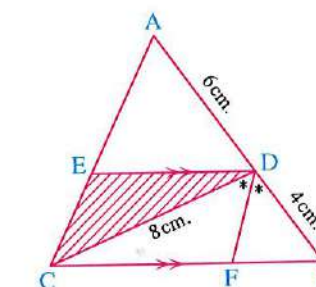


- (8) In the opposite figure :

If the area of $(\triangle DBF) = 10$ cm²

, then the area of $(\triangle DEC) = \dots\dots\dots$ cm²

- (a) 12 (b) 16
(c) 18 (d) 24



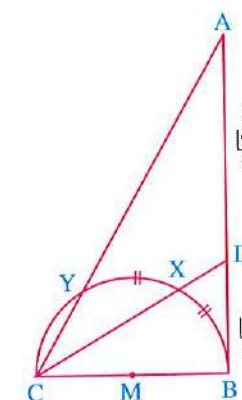
- (9) In the opposite figure :

If $m(\widehat{BX}) = m(\widehat{XY})$

, $BD = 2\sqrt{3}$ cm. , $AD = 4\sqrt{3}$ cm.

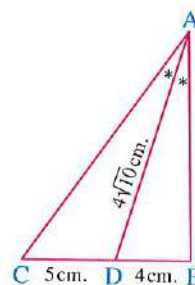
, then $AY = \dots\dots\dots$ cm.

- (a) $4\sqrt{3}$ (b) 6
(c) 9 (d) 12

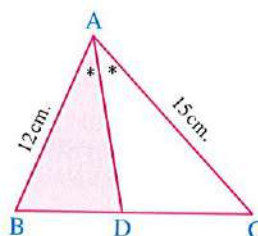


(10) In the opposite figure :The perimeter of $\triangle ABC = \dots\dots\dots$ cm.

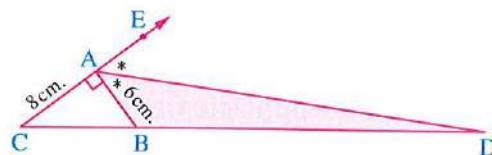
- (a) 36 (b) 32
(c) 28 (d) 24

**(11) In the opposite figure :**If the area of $(\triangle ABC) = 72 \text{ cm}^2$, then the area of $(\triangle ADB) = \dots\dots\dots \text{cm}^2$

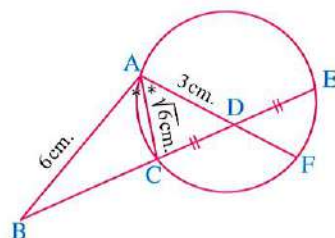
- (a) 24 (b) 28
(c) 32 (d) 40

**(12) In the opposite figure :**The area of $(\triangle ABD) = \dots\dots\dots \text{cm}^2$

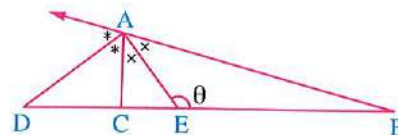
- (a) 36 (b) 48
(c) 54 (d) 72

**(13) In the opposite figure :** \overrightarrow{AC} bisects $\angle BAD$, D is the midpoint of \overline{EC} , $AC = \sqrt{6} \text{ cm}$, $AD = 3 \text{ cm}$., $AB = 6 \text{ cm}$, then $DF = \dots\dots\dots \text{cm}$.

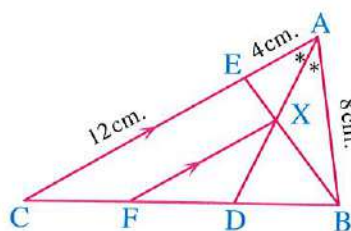
- (a) 2 (b) 3
(c) 3.5 (d) 4

**(14) In the opposite figure :**If $AD = 8 \text{ cm}$, $AE = 6 \text{ cm}$, then $\tan \theta = \dots\dots\dots$

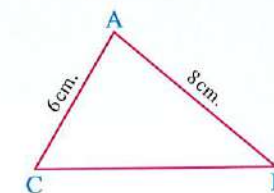
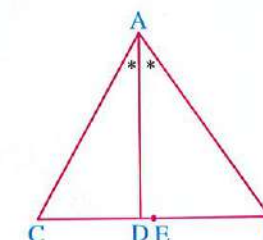
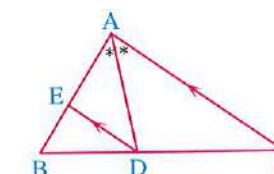
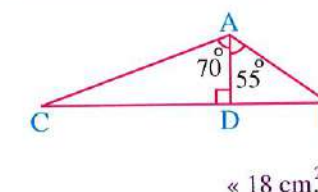
- (a) $\frac{-4}{3}$ (b) $\frac{-3}{4}$
(c) $\frac{3}{4}$ (d) $\frac{4}{3}$

**(15) In the opposite figure :** $\frac{DF}{BC} = \dots\dots\dots$

- (a) $\frac{4}{3}$ (b) $\frac{2}{3}$
(c) $\frac{3}{5}$ (d) $\frac{1}{3}$

**(16) In the opposite figure :**If $m(\angle A) = 2 m(\angle B)$, then $BC = \dots\dots\dots \text{cm}$.

- (a) $3\sqrt{10}$ (b) $2\sqrt{21}$
(c) 12 (d) 10

**34 In the opposite figure :**ABC is a triangle in which : $AB > AC$, E is the midpoint of \overline{BC} , \overrightarrow{AD} bisects $\angle A$ internally.Prove that : $\frac{ED}{EC} = \frac{AB - AC}{AB + AC}$ **35 In the opposite figure :**ABC is a triangle, \overrightarrow{AD} bisects $\angle BAC$ internally, $\overline{DE} \parallel \overline{AC}$ and intersects \overline{AB} at EProve that : $DE = \frac{AB \times AC}{AB + AC}$ **36 In the opposite figure :**If $AC \times BD = 36 \text{ cm}^2$ Find the area of $(\triangle ABC)$ 

Follow : angle bisector and proportional parts (Converse of theorem 3)

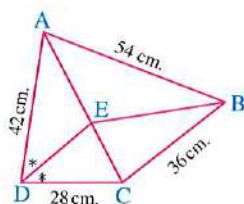
From the school book

- 1 ABC is a triangle in which : $AB = 6$ cm. , $AC = 9$ cm. , $BC = 10.5$ cm. , $D \in \overline{BC}$, where $BD = 4.2$ cm. **Prove that :** \overrightarrow{AD} bisects $\angle BAC$

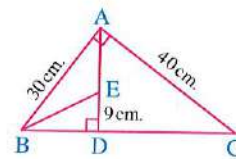
- 2 ABC is a triangle in which $AB = 6$ cm. , $BC = 4$ cm. , $CA = 3.6$ cm. , $D \in \overline{BC}$ such that $CD = 6$ cm. **Prove that :** \overrightarrow{AD} bisects the exterior angle of $\triangle ABC$ at A

- 3 In each of the following figures , prove that : \overrightarrow{BE} bisects $\angle ABC$

(1)



(2)



- 4 ABCD is a quadrilateral in which $AB = 6$ cm. , $BC = 9$ cm. , $CD = 6$ cm. , $AD = 4$ cm. , \overrightarrow{AE} bisects $\angle A$ and intersects \overline{BD} at E

(1) Find the value of the ratio : $\frac{BE}{ED}$

(2) Prove that : \overrightarrow{CE} bisects $\angle BCD$

« $\frac{3}{2}$ »

- 5 ABCD is a quadrilateral in which $AB = 18$ cm. , $BC = 12$ cm. , $E \in \overline{AD}$, where $2AE = 3ED$, draw $\overrightarrow{EF} \parallel \overline{DC}$ and intersects \overline{AC} at F
- Prove that :** \overrightarrow{BF} bisects $\angle ABC$

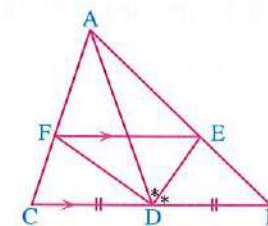
- 6 In the opposite figure :

D is the midpoint of \overline{BC} ,

\overrightarrow{DE} bisects $\angle ADB$, $\overrightarrow{EF} \parallel \overline{BC}$

Prove that : (1) \overrightarrow{DF} bisects $\angle ADC$

(2) $\overline{ED} \perp \overline{DF}$



- 7 ABC is a triangle , X is the midpoint of \overline{BC} , $BX = 6$ cm. , $AX = 9$ cm. , the bisector of $\angle AXB$ intersects \overline{AB} at D , take $E \in \overline{AC}$, where $AE = 6$ cm. given that $AC = 10$ cm.

(1) Find the value of : $\frac{AD}{DB}$

« $\frac{3}{2}$ »

(2) Prove that : $\overline{DE} \parallel \overline{BC}$

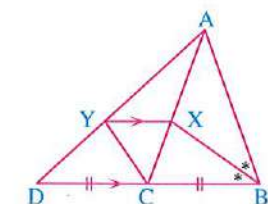
(3) Prove that : \overrightarrow{XE} bisects $\angle AXC$

- 8 In the opposite figure :

$AB = AC$, $BC = CD$,

\overrightarrow{BX} bisects $\angle ABC$, $\overrightarrow{XY} \parallel \overline{BD}$

Prove that : \overrightarrow{CY} bisects $\angle ACD$

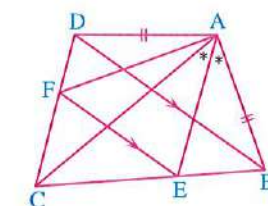


- 9 In the opposite figure :

$AB = AD$, \overrightarrow{AE} bisects $\angle BAC$,

$\overline{EF} \parallel \overline{BD}$

Prove that : \overrightarrow{AF} bisects $\angle CAD$



- 10 ABC is a triangle , $D \in \overline{BC}$, $D \notin \overline{BC}$, where $CD = AB$, draw $\overrightarrow{CE} \parallel \overline{DA}$ and intersects \overline{AB} at E , draw $\overrightarrow{EF} \parallel \overline{BC}$ and intersects \overline{AC} at F

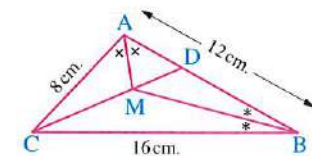
Prove that : \overrightarrow{BF} bisects $\angle ABC$

- 11 In the opposite figure :

ABC is a triangle in which $AB = 12$ cm. ,

$AC = 8$ cm. , $BC = 16$ cm. , \overrightarrow{BM} bisects $\angle ABC$,

\overrightarrow{AM} bisects $\angle BAC$ Find : The length of \overline{AD}



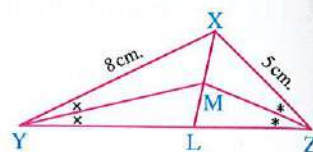
« 4 cm. »

12 In the opposite figure :

\overrightarrow{ZM} and \overrightarrow{YM} bisect $\angle Z$ and $\angle Y$ respectively

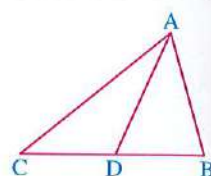
, $XY = 8$ cm. , $XZ = 5$ cm.

Prove that : $8LZ = 5LY$

**13 In the opposite figure :**

If $AC : CD : AB : BD = 15 : 10 : 9 : 6$,

Prove that : \overrightarrow{AD} bisects $\angle BAC$

**14 ABC is a triangle in which $AB = 5$ cm. , $AC = 10$ cm. , $BC = 9$ cm. , $D \in \overline{BC}$**

such that $BD = 3$ cm. , $E \in \overline{CB}$, where $\overline{AE} \perp \overline{AD}$

(1) Prove that : \overrightarrow{AD} bisects $\angle BAC$

(2) Find : The length of \overline{BE}

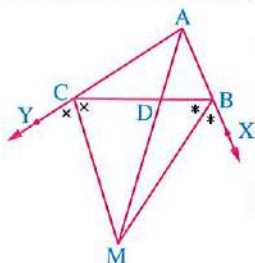
« 9 cm. »

15 In the opposite figure :

\overrightarrow{BM} bisects $\angle CBX$,

\overrightarrow{CM} bisects $\angle BCY$

Prove that : \overrightarrow{AM} bisects $\angle BAC$

**16 ABC is a triangle in which $AB = 6$ cm. , $BC = 12$ cm. , $CA = 9$ cm. , $D \in \overline{AB}$, where**

$AD = 2$ cm. , draw $\overrightarrow{DE} \parallel \overline{BC}$ and intersects \overline{AC} at E , find the length of \overline{AE} , then

prove that : \overrightarrow{BE} bisects $\angle ABC$

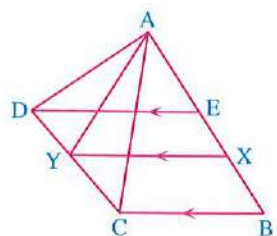
« 3 cm. »

17 In the opposite figure :

$\overline{ED} \parallel \overline{XY} \parallel \overline{BC}$

and $AD \times BX = AC \times EX$

Prove that : \overrightarrow{AY} bisects $\angle CAD$

**18 Two circles M and N are touching externally at A , a straight line is drawn parallel to**

\overline{MN} and intersects the circle M at B , C and the circle N at D , E respectively.

If $\overline{BM} \cap \overline{EN} = \{F\}$, **prove that :** \overrightarrow{FA} bisects $\angle MFN$

19 \overline{AB} is a diameter of a circle , \overline{AC} is a chord in it , \overline{CD} is a tangent drawn to the circle at C and intersects \overline{AB} at D. If $E \in \overline{AB}$, where $\frac{DB}{BE} = \frac{DC}{CE}$

Prove that : (1) \overrightarrow{CA} bisects the exterior angle of $\triangle CDE$ at C

(2) $\frac{DA}{DB} = \frac{AE}{BE}$

**Problems that measure high standard levels of thinking****20 In the opposite figure :**

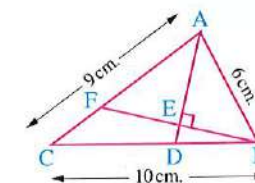
ABC is a triangle in which $AB = 6$ cm. , $AC = 9$ cm. ,

and $BC = 10$ cm. , $D \in \overline{BC}$, where $BD = 4$ cm.

$\overrightarrow{BE} \perp \overline{AD}$ and intersects \overline{AD} and \overline{AC} at E and F respectively.

(1) Prove that : \overrightarrow{AD} bisects $\angle A$

(2) Find : Area of $\triangle ABF$: area of $\triangle CBF$



« 2 »

Applications of
proportionality in the circleTest
yourself

From the school book

1 Find the power of the given point with respect to the circle M whose radius length is r :

- (1) The point A where $AM = 12$ cm. and $r = 9$ cm.
 (2) The point B where $BM = 8$ cm. and $r = 15$ cm.
 (3) The point C where $CM = 7$ cm. and $r = 7$ cm.
 (4) The point D where $DM = \sqrt{17}$ cm. and $r = 4$ cm.

2 Determine the position of each of the following points with respect to the circle M , of radius length 10 cm., then calculate the distance between each point and the centre of the circle :

- (1) $P_M(A) = -36$ (2) $P_M(B) = 96$ (3) $P_M(C) = \text{zero}$

3 If the distance between a point and the centre of a circle equals 25 cm., and the power of this point with respect to the circle equals 400, find the radius length of this circle.

« 15 cm. »

4 If a point A is outside the circle M , \overline{AD} is a tangent to the circle at D where $AD = 8$ cm., find the power of point A with respect to circle M

« 64 »

5 Choose the correct answer from those given :

- (1) If the power of a point A with respect to the circle M is a negative quantity, then A lies
 (a) inside the circle. (b) on the centre of the circle.
 (c) outside the circle. (d) on the circle.
 (2) If M is a circle, A is a point that lies in its plane where $P_M(A) = 0$, then A lies
 (a) inside the circle. (b) on the centre of the circle.
 (c) outside the circle. (d) on the circle.

(3) If M is a circle of radius length 3 cm., A is a point lies in its plane where $MA = 4$ cm., then $P_M(A) = \dots\dots\dots$

- (a) $\sqrt{7}$ (b) 9 (c) 7 (d) -7

(4) If N is a circle of diameter length 16 cm., B is a point lies in its plane where $NB = 5$ cm., then $P_N(B) = \dots\dots\dots$

- (a) 39 (b) -39 (c) $\sqrt{39}$ (d) -231

(5) If the power of a point with respect to circle M equals -625, the distance between this point and the centre of the circle = 15 cm., then the diameter length of this circle equals cm.

- (a) 400 (b) 20 (c) $5\sqrt{34}$ (d) $10\sqrt{34}$

(6) If M is a circle, A is a point in its plane where $MA = 6$ cm., $P_M(A) = -13$, then area of this circle = ($\pi = \frac{22}{7}$)

- (a) 154 (b) 44 (c) 144 (d) 7

(7) If M is a circle of radius length 7 cm., A is a point in its plane, 25 cm. apart from the centre of the circle, then the length of the tangent segment to the circle M from A is cm.

- (a) 5 (b) 49 (c) 24 (d) 12

6 In the opposite figure :

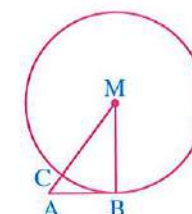
\overline{AB} is a tangent to the circle M at B

\overline{MA} intersects the circle M at C

If the radius length of the circle equals 12 cm.

$P_M(A) = 81$, then find :

- (1) The length of \overline{AB} (2) The length of \overline{AC} « 9 cm., 3 cm. »



7 The radius length of circle M equals 31 cm. The point A lies at 23 cm. distant from its centre. Draw the chord \overline{BC} where $A \in \overline{BC}$, $AB = 3 AC$ Calculate :

- (1) The length of the chord \overline{BC}
 (2) The distance between the chord \overline{BC} and the centre of the circle. « 48 cm., 19.6 cm. »

8 The radius length of circle N equals 8 cm. The point B lies at 12 cm. distant from its centre, draw a straight line passes through the point B and intersects the circle at C and D where $CB = CD$ Calculate the length of the chord \overline{CD} and its distance from the point N

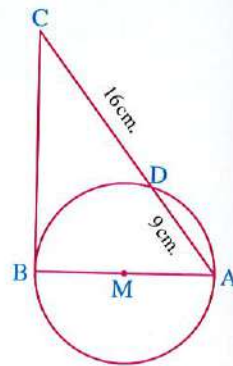
« $2\sqrt{10}$ cm., $3\sqrt{6}$ cm. »

9 In the opposite figure :

M is a circle, \overline{AB} is a diameter in it
 \overline{CB} is a tangent to the circle M at B
 \overline{CA} intersects the circle M at D, where
 $CD = 16$ cm. , $DA = 9$ cm. **Find :**

(1) The length of the circle's radius.

(2) The area of triangle ABC



« 7.5 cm. , 150 cm². »

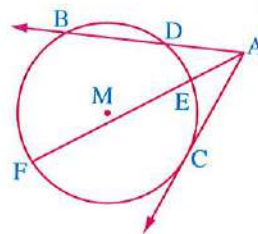
10 In the opposite figure :

A is a point outside the circle M, \overline{AB} intersects
the circle at D, B, \overline{AF} intersects the circle at E, F,
 \overline{AC} is a tangent to the circle at C,
 $AD = 8$ cm. , $EF = 18$ cm.

(1) If $P_M(A) = 144$, find the length of each of : \overline{AC} , \overline{DB} , \overline{AE}

(2) If $X \in \overline{BD}$ where $DX = 4$ cm. , find : $P_M(X)$

« 12 cm. , 10 cm. , 6 cm. , - 24 »



11 The two circles M and N are touching each other externally at A, \overline{AB} is a common
tangent to the two circles M, N. \overline{BC} intersects the circle M at C and D. \overline{BE} intersects the
circle N at E and F respectively.

(1) Prove that : \overline{AB} is the principle axis of the two circles M and N

(2) If $P_M(B) = 36$, $BC = 4$ cm. , $EF = 9$ cm.

Find the length of each of : \overline{CD} , \overline{AB} and \overline{BE}

« 5 cm. , 6 cm. , 3 cm. »

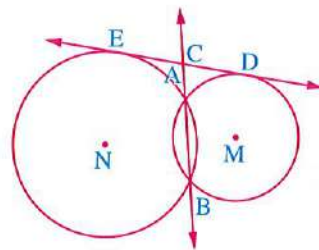
12 In the opposite figure :

M, N are two intersecting circles at A, B
 \overline{ED} is a common tangent to the two circles M, N
at D, E respectively. $\overline{AB} \cap \overline{DE} = \{C\}$

(1) Prove that : \overline{BC} is the principle axis of the two circles.

(2) If $AB = 12$ cm. , $P_N(C) = 64$, find the length of each of : \overline{CA} , \overline{CD}

« 4 cm. , 8 cm. »

**13 In the opposite figure :**

The two circles M and N are intersecting at

A and B where $\overline{AB} \cap \overline{CD} \cap \overline{EF} = \{X\}$,

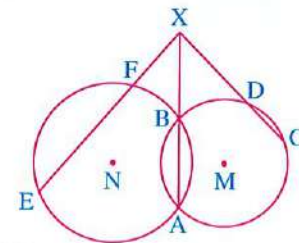
$XD = 2 DC$, $EF = 10$ cm. and $P_N(X) = 144$

(1) Prove that : \overline{AB} is the principle axis to the two circles M and N

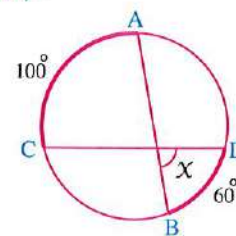
(2) Find the length of each of : \overline{XC} and \overline{XF}

(3) Prove that : CDFE is a cyclic quadrilateral.

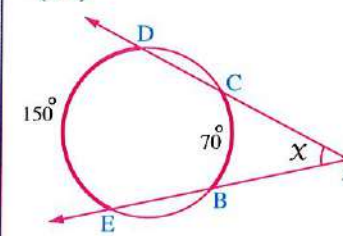
« $6\sqrt{6}$ cm. , 8 cm. »

**14 Using the given data in each figure , find the value of the symbol used in measurement :**

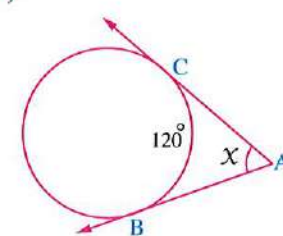
(1)



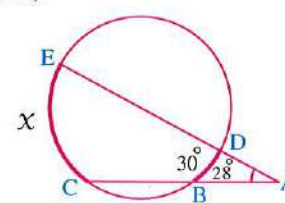
(2)



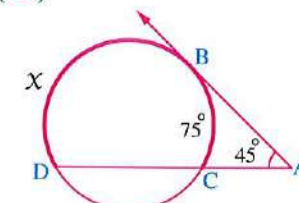
(3)



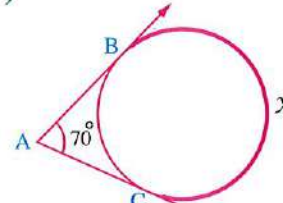
(4)



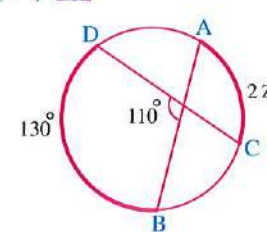
(5)



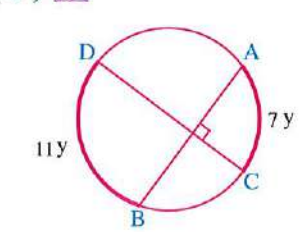
(6)



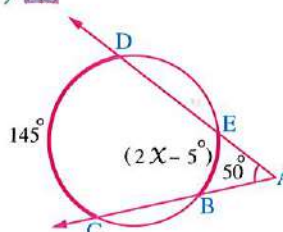
(7)



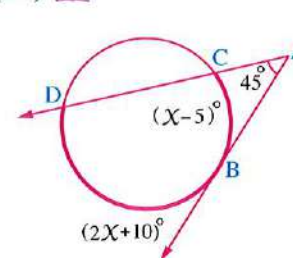
(8)



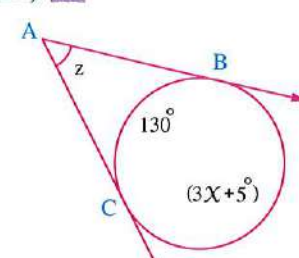
(9)



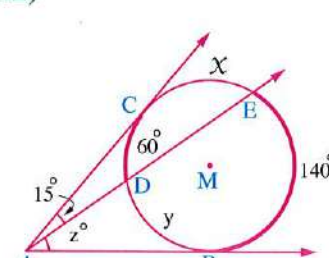
(10)



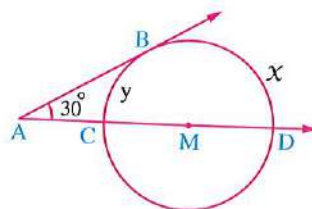
(11)



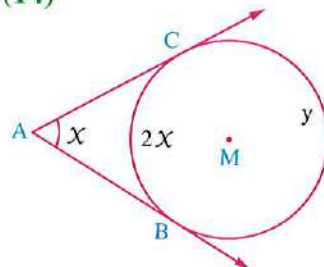
(12)



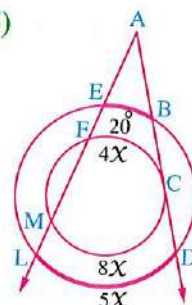
(13)



(14)



(15)



15 In the opposite figure :

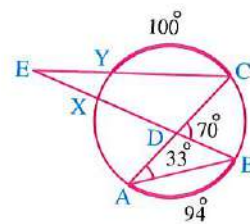
$$m(\angle BAC) = 33^\circ, m(\angle BDC) = 70^\circ,$$

$$m(\widehat{AB}) = 94^\circ, m(\widehat{CY}) = 100^\circ \text{ Find the measure of each of :}$$

(1) \widehat{XY}

(2) \widehat{AX}

(3) $\angle BEC$



$$\ll 26^\circ, 74^\circ, 20^\circ \gg$$

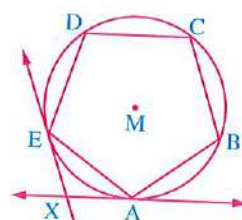
16 In the opposite figure :

ABCDE is a regular pentagon drawn inside the circle M ,

\overrightarrow{AX} is a tangent to the circle at A , \overrightarrow{EX} is a tangent to the circle at E where $\overrightarrow{AX} \cap \overrightarrow{EX} = \{X\}$ Find :

(1) $m(\widehat{AE})$

(2) $m(\angle AXE)$



$$\ll 72^\circ, 108^\circ \gg$$

Problems that measure high standard levels of thinking

17 Choose the correct answer from those given :

(1) In the opposite figure :

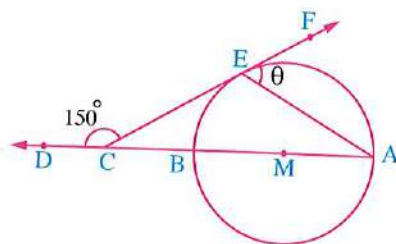
$$\theta = \dots\dots\dots$$

(a) 45°

(b) 50°

(c) 55°

(d) 60°



(2) In the opposite figure :

If $AE = AB$, \overline{BC} is a diameter , $m(\angle D) = 21^\circ$

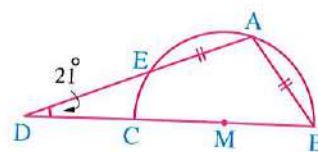
, then $m(\angle A) = \dots\dots\dots$

(a) 100°

(b) 104°

(c) 106°

(d) 110°



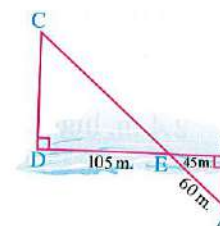
Life Applications on Unit Four

From the school book

1 To determine the location C ,

surveyors measure and prepare the opposite scheme.

Find the distance between the location C and the location A

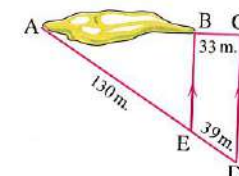


$$\ll 200 \text{ m.} \gg$$

2 A team of pollution control determined

the location of an oil spot on one of the beaches as in the opposite figure.

Calculate the length of the oil spot.



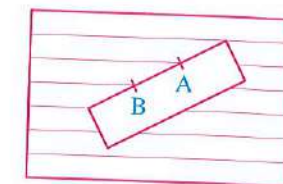
$$\ll 110 \text{ m.} \gg$$

3 Yousef wanted to divide a strip of paper

into 3 equal parts in length. He placed it on a paper on his notebook , as in the opposite figure , and determined two points of division A and B

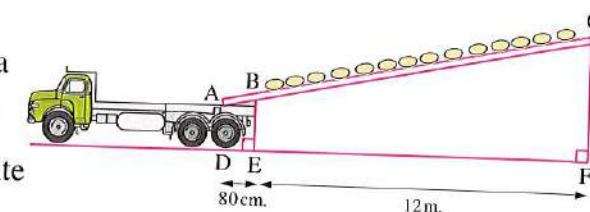
Is the division of Yousef's strip correct ? Explain your answer.

Use your geometric instruments to verify your answer.



4 Fertilizer packages produced from one

of the factories are transferred by sliding on a tube that is inclined and carried on to trucks to the centre of distributions as in the opposite figure.

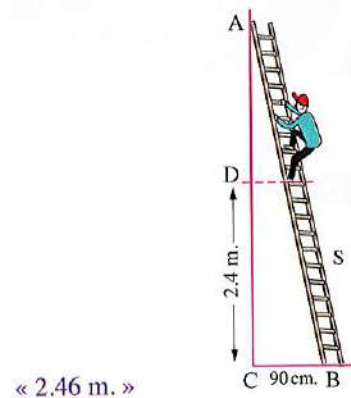


If D , E and F are the projections of the points A , B and C on the horizontal respectively , $AB = 1.2 \text{ m.}$, $DE = 80 \text{ cm.}$, $EF = 12 \text{ m.}$

Find the length of the tube to the nearest metre.

$$\ll 19 \text{ m.} \gg$$

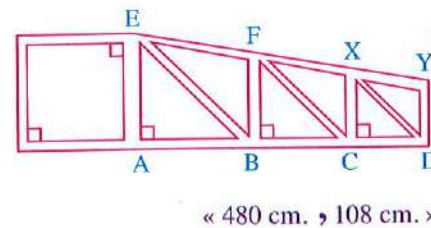
- 5 \overline{AB} is a ladder of length 4.1 metres rests by its upper end A on a vertical wall and with its lower end B on a horizontal rough ground. If the lower end is 90 cm. apart from the wall, calculate the distance which a man ascends on the ladder until it becomes at 2.4 m. high from the ground.



- 6 If $AB = 180$ cm. , $EF = 2$ m. ,

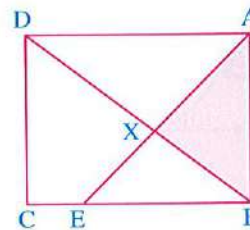
$$AB : BC : CD = 5 : 4 : 3$$

Find the length of each of : \overline{EY} and \overline{CD}

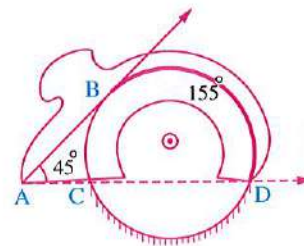


- 7 The opposite figure shows a rectangular piece of land divided into four different parts by the two lines \overrightarrow{BD} and \overrightarrow{AE} , where $E \in \overline{BC}$, $\overrightarrow{BD} \cap \overrightarrow{AE} = \{X\}$, if $AB = BE = 42$ metres , $AD = 56$ metres

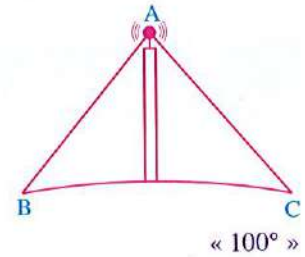
Calculate the area of the piece ABX in square metres and the length of \overline{AX}



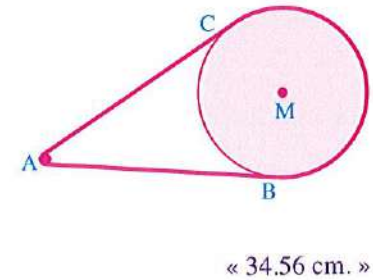
- 8 A circular saw for cutting wood, the radius length of its circle equals 10 cm. It rotates inside a protective container. If $m(\angle BAD) = 45^\circ$ and $m(\widehat{BD}) = 155^\circ$
Find the arc length of the disc's saw outside the protective container.



- 9 The signals produced from the communication tower follow a ray in their pathway, its starting point is on the top of the tower and it is a tangent to the surface of the earth, as in the opposite figure. Determine the measure of the arc included by the two tangents supposing that the tower lies at sea level and $m(\angle CAB) = 80^\circ$



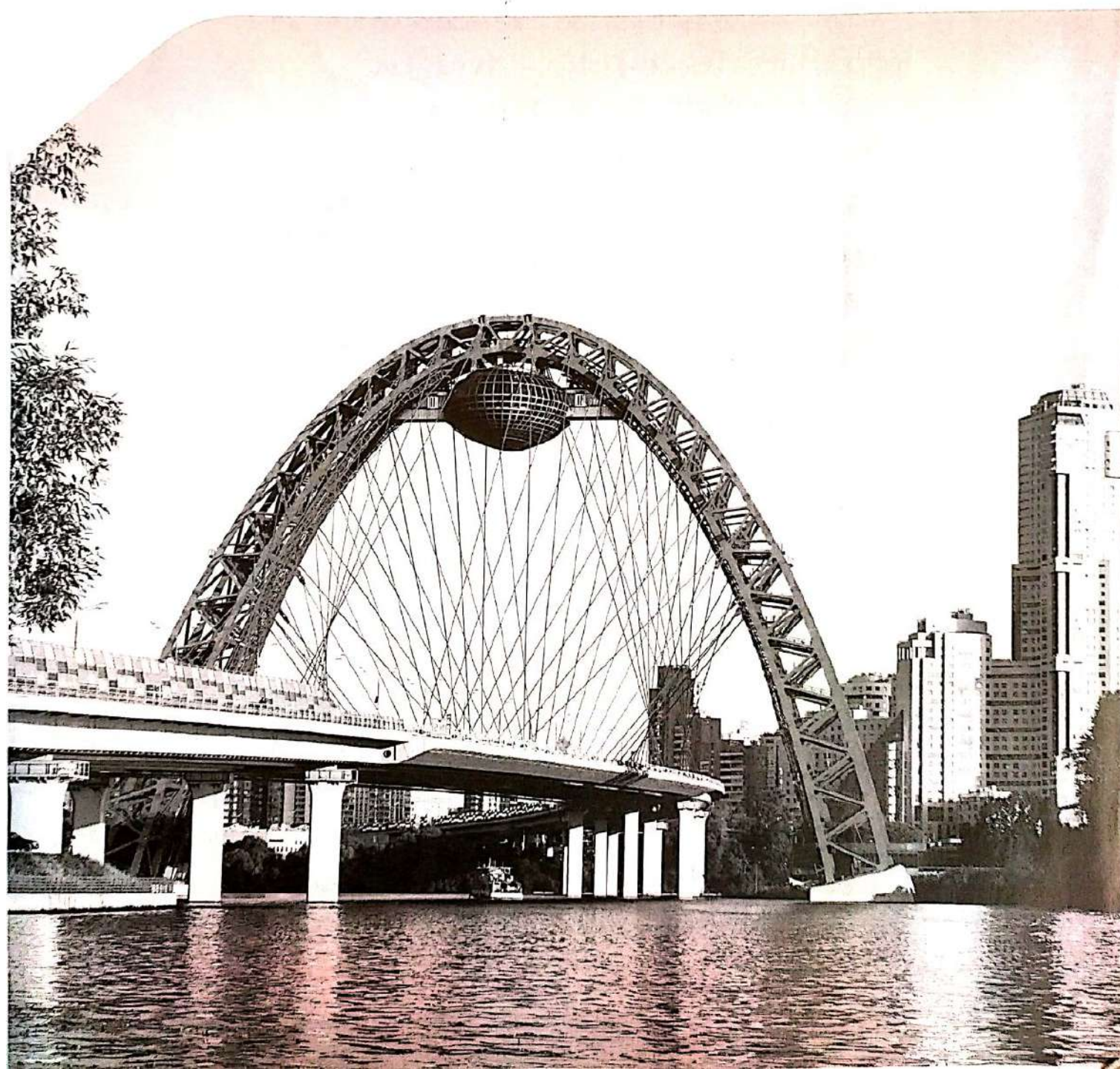
- 10 A pulley rotates at the axis M by a strap passing over a small pulley at A. If the measure of the angle between the two parts of the strap is 40° Find the length of the major arc \widehat{BC} , given that the radius length of the larger pulley equals 9 cm.



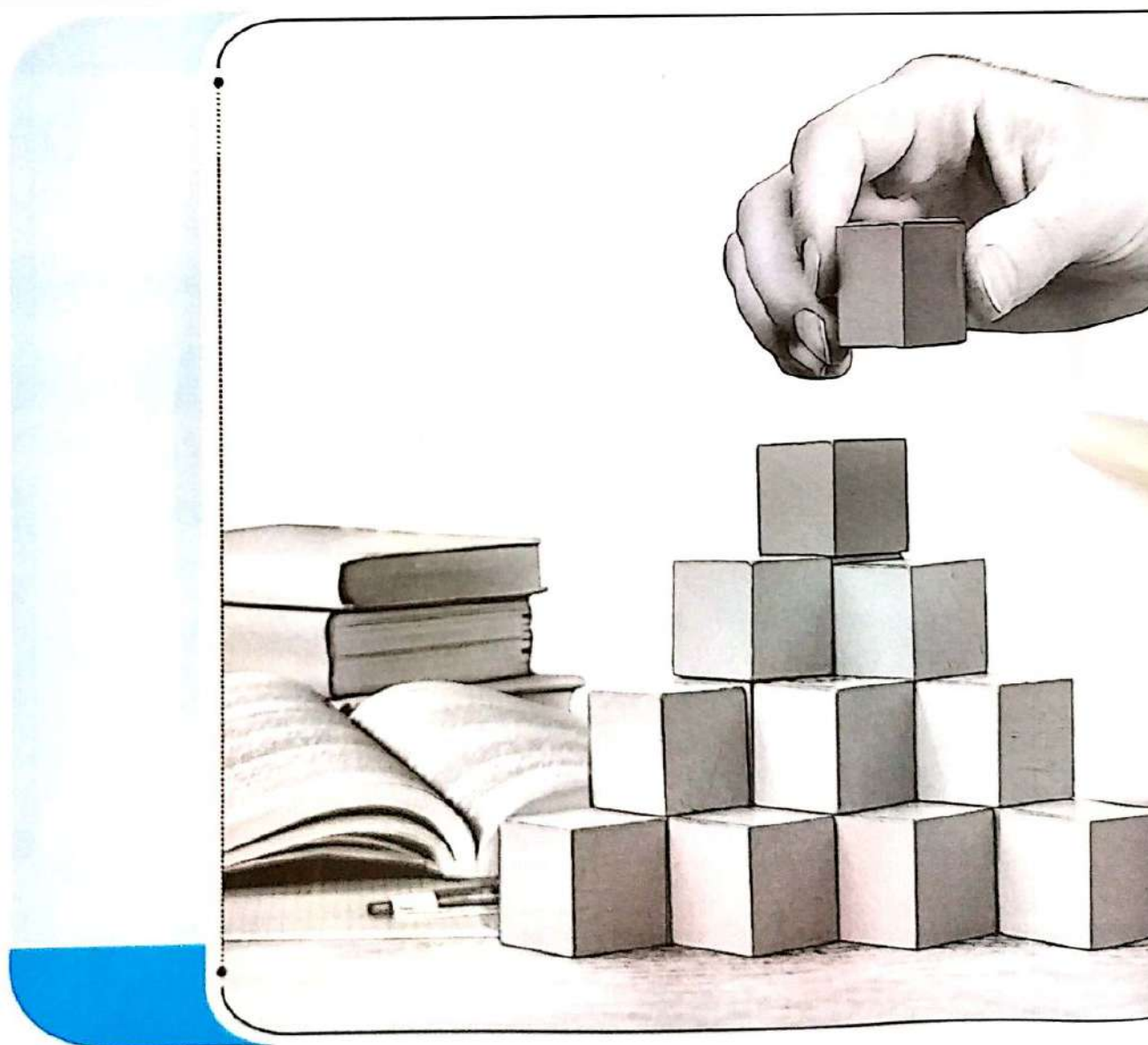
- 11 A satellite revolves in an orbit and keeps in during rotation on a fixed height above the equator. The camera on it can monitor the arc length of 6011 km. on the surface of the earth. If the measure of the arc equals 54° , find :
- (1) The measure of the angle of the camera placed on the satellite.
 - (2) The radius length of the Earth of the equator.
- « 126° , 6378 km. »

First

Algebra and Trigonometry



Accumulative Tests on Algebra and Trigonometry



First : Accumulative Tests on Algebra

Test

1

on lesson 1 - unit 1

Total mark

10

Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) If $X = 4$ is one of the two roots of the equation : $X^2 + mX = 4$, then $m = \dots\dots\dots$

- (a) 3 (b) -3 (c) -4 (d) 1

(2) The solution set of the equation : $X^2 + 5 = 0$ in \mathbb{R} is $\dots\dots\dots$

- (a) $\{5\}$ (b) $\{\sqrt{5}\}$ (c) $\{\sqrt{5}, -\sqrt{5}\}$ (d) \emptyset

(3) The solution set of the equation : $X^2 - X - 72 = 0$ in \mathbb{R} is $\dots\dots\dots$

- (a) $\{8, 9\}$ (b) $\{-8, 9\}$ (c) $\{-8, -9\}$ (d) $\{8, -9\}$

(4) The solution set of the equation : $(X - 3)^2 = (X - 3)$ in \mathbb{R} is $\dots\dots\dots$

- (a) $\{3\}$ (b) $\{4\}$ (c) $\{-3, -4\}$ (d) $\{3, 4\}$

Second question

3 marks

each figure 1 mark

Each of the following figures shows the graphical representation of the second degree function f : Find in \mathbb{R} the solution set of the equation $f(X) = 0$ in each figure :

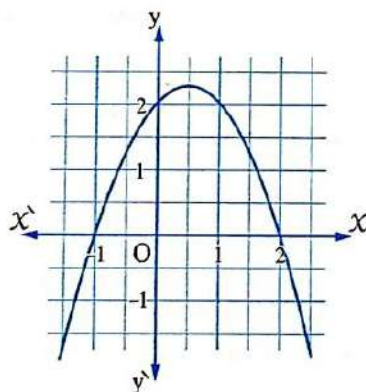


Figure (1)

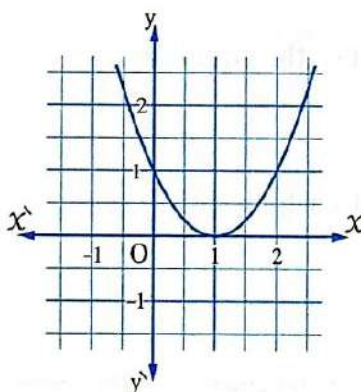


Figure (2)

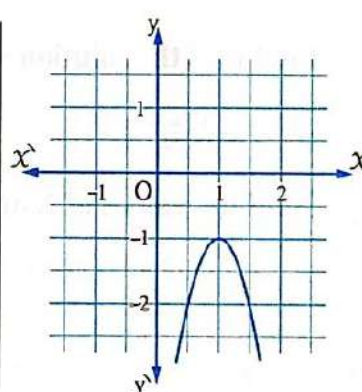


Figure (3)

Third question

5 marks

* Graphically 2.5 marks. * Algebraically 2.5 marks.

Find graphically in \mathbb{R} the solution set of the following equation ,
then verify the result algebraically : $2X(X + 2) = 1$

Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) $\sqrt{-2} \times \sqrt{-8} = \dots\dots\dots$

(a) 4

(b) -4

(c) $4i$

(d) -16

(2) The simplest form of the imaginary number i^{42} is $\dots\dots\dots$

(a) -1

(b) 1

(c) i (d) $-i$

(3) The solution set of the equation : $x^2 + 9 = 0$ in \mathbb{C} is $\dots\dots\dots$

(a) $\{3, -3\}$ (b) $\{-3i\}$ (c) $\{3i, -3i\}$ (d) \emptyset

(4) If the curve of the quadratic function f intersects the x -axis at the two points $(3, 0)$, $(-1, 0)$, then the solution set of the equation : $f(x) = 0$ in \mathbb{R} is $\dots\dots\dots$

(a) $\{3, 0\}$ (b) $\{-1, 0\}$ (c) $\{-3, 1\}$ (d) $\{3, -1\}$

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] Find in \mathbb{C} the solution set of the equation :

$$x^2 - 2x + 4 = 0$$

[b] Find the values of x and y which satisfy that :

$$x + iy = \frac{(2+i)(2-i)}{3+2i}$$

Third question

4 marks

[a] 2 marks

[b] 2 marks

[a] Find graphically in \mathbb{R} the solution set of the equation : $x^2 - 4x + 3 = 0$ in the interval $[-1, 5]$

[b] Put in the simplest form each of :

(1) $\frac{5}{1-2i}$

(2) $\frac{2+3i}{2-3i}$

**Test****3****till lesson 3 – unit 1****Total mark****10**

Answer the following questions :

First question**2 marks**each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) If the two roots of the equation : $4x^2 - 12x + c = 0$ are equal , then $c = \dots\dots\dots$

- (a) 3 (b) 4 (c) 9 (d) 16

(2) If $x = -1$ is one of the roots of the equation : $x^2 - ax - 2 = 0$, then $a = \dots\dots\dots$

- (a) 1 (b) -1 (c) 3 (d) -3

(3) If $a = 1 + \sqrt{2}i$, $b = 1 - \sqrt{2}i$, then $ab = \dots\dots\dots$

- (a) -1 (b) 1 (c) 2 (d) 3

(4) If the two roots of the equation : $x^2 - 6x + k = 0$ are different and real

, then $k \in \dots\dots\dots$

- (a) $]-\infty, 9[$ (b) $]9, \infty[$ (c) $]-\infty, 9]$ (d) $[9, \infty[$

Second question**4 marks**

[a] 2 marks

[b] 2 marks

[a] Prove that the two roots of the equation : $3x^2 - 4x + 5 = 0$ are not real ,

then find the solution set of the equation in \mathbb{C}

[b] Find the values of k which make the equation : $kx^2 - 4x + 4 = 0$ have two

complex and not real roots.

Third question**4 marks**

[a] 2 marks

[b] 2 marks

[a] Put in the simplest form : $\frac{(3+i)(3-i)}{3-4i}$

[b] Find the values of x and y which satisfy the equation : $x + iy = \frac{2-3i}{i}$



Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

- (1) If one of the two roots of the equation : $x^2 - (m - 3)x + 5 = 0$ is the additive inverse of the other root , then $m = \dots\dots\dots$
(a) - 5 (b) - 3 (c) 3 (d) 5
- (2) The simplest form of the imaginary number i^{31} is $\dots\dots\dots$
(a) i (b) $-i$ (c) 1 (d) - 1
- (3) If one of the two roots of the equation : $ax^2 + 2x + 5 = 0$ is the multiplicative inverse of the other root , then $a = \dots\dots\dots$
(a) - 5 (b) - 2 (c) 2 (d) 5
- (4) If the two roots of the equation : $x^2 + 4x + k = 0$ are real , then $k \in \dots\dots\dots$
(a) $[4, \infty[$ (b) $]4, \infty[$ (c) $] - \infty, 4]$ (d) $] - \infty, 4[$

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] If $x = 3 + 2i$, $y = \frac{4 - 2i}{1 - i}$

, find $x + y$ in the form of a complex number.

[b] If the product of the two roots of the equation : $2x^2 + 6x + c = 0$ equals $\frac{5}{2}$

, find the value of c , then find the solution set of the equation in the set of complex numbers.

Third question

4 marks

[a] 2 marks

[b] 2 marks

[a] If the two roots of the equation : $x^2 - 3x + 2 + \frac{1}{m} = 0$ are equal

, find the value of : m

[b] Find the value of k which makes one of the two roots of the equation :

$x^2 + 3x + k = 0$ double the other root.



Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) The solution set of the equation : $X^2 - 4X = -4$ in \mathbb{R} is

- (a) $\{-2\}$ (b) $\{2\}$ (c) $\{-2, 2\}$ (d) \emptyset

(2) The quadratic equation whose roots are $i, -i$ is

- (a) $X^2 - 1 = 0$ (b) $X^2 + 1 = 0$ (c) $(X + 1)^2 = 0$ (d) $(X - 1)^2 = 0$

(3) The two roots of the equation : $X^2 - 2X + k = 0$ are real and different if

- (a) $k = 1$ (b) $k < 1$ (c) $k > 1$ (d) $k = 4$

(4) The simplest form of the expression : $(1 - i)^4$ is

- (a) -4 (b) 4 (c) $-4i$ (d) $4i$

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] If L, M are the two roots of the equation : $2X^2 + 2X + 3 = 0$,

find the equation whose two roots are : $\frac{2}{L}, \frac{2}{M}$

[b] Find the simplest form of the expression : $(3 - 2i)^2 (3 + 2i)$

Third question

4 marks

[a] 2 marks

[b] 2 marks

[a] If $2 + i$ is one of the two roots of the equation : $X^2 - 4X + c = 0$ where $c \in \mathbb{R}$

, find the other root , then find the value of c

[b] Determine the type of the two roots of the equation : $X^2 + 9 = 6X$

, then find the solution set in \mathbb{R}



Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

- (1) The function $f : [-2, 4] \longrightarrow \mathbb{R}$, $f(x) = 4 - 2x$ is negative in the interval
- (a) $[-2, 0[$ (b) $]0, 4]$ (c) $[2, 4]$ (d) $]2, 4]$
- (2) If the two roots of the equation : $x^2 - 6x + k = 0$ are equal , then $k = \dots\dots\dots$
- (a) 9 (b) 6 (c) 1 (d) 12
- (3) The quadratic equation whose two roots are $(1 + i)$, $(1 - i)$ is
- (a) $x^2 - 2x + 2 = 0$ (b) $x^2 + 2x - 2 = 0$
 (c) $x^2 + 2x + 2 = 0$ (d) $x^2 - 2x - 2 = 0$
- (4) If one of the two roots of the equation : $ax^2 - 3x + 2 = 0$ is the multiplicative inverse of the other root , then $a = \dots\dots\dots$
- (a) $\frac{1}{2}$ (b) 3 (c) 2 (d) -2

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] If L, M are the two roots of the equation : $x^2 - 2x - 8 = 0$

, find the equation whose two roots are : $\frac{1}{L}, \frac{1}{M}$

[b] If $x = \frac{13 + 13i}{5 + i}$, $y = \frac{5 + i}{1 + i}$, prove that : x, y are two conjugate numbers ,

then find the value of : $x^2 + 2xy + y^2$

Third question

4 marks

(1) 2 marks

(2) 2 marks

Determine the sign of each of the two functions defined by the following rules , representing your answer on the number line :

(1) $f(x) = (x - 1)(x + 2)$

(2) $f(x) = -x^2 + 9$



Test

7

till lesson 7 – unit 1

Total mark

10

Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) The function $f : f(x) = -3$ is negative in

- (a) $]-\infty, -3]$ (b) $]-3, 3[$ (c) $]-\infty, \infty[$ (d) $]-\infty, 0[$

(2) The solution set of the inequality : $x(x-2) \geq 0$ in \mathbb{R} is

- (a) $\{0, 2\}$ (b) $[0, 2]$ (c) $\mathbb{R} - [0, 2]$ (d) $\mathbb{R} -]0, 2[$

(3) The simplest form of the imaginary number i^{52} is

- (a) i (b) $-i$ (c) 1 (d) -1

(4) If one of the two roots of the equation : $ax^2 + 4x + 7 = 0$ is the multiplicative inverse of the other root , then $a =$

- (a) $\frac{1}{7}$ (b) 7 (c) 4 (d) -7

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] If $1 + i$ is one of the two roots of the equation : $x^2 - 2x + c = 0$ where $c \in \mathbb{R}$, find the other root , then find the value of c

[b] Investigate the sign of the function $f : f(x) = 2x^2 + 7x - 15$ and from this find in \mathbb{R} the solution set of the inequality : $2x^2 + 7x \leq 15$

Third question

4 marks

[a] 2 marks

[b] 2 marks

[a] If $\frac{1}{L}, \frac{1}{M}$ are the two roots of the equation : $x^2 - 4x + 2 = 0$, form the equation whose roots are L, M

[b] If $a + bi = \frac{8+6i}{1+i}$, find the real values of : a, b

Second : Accumulative Tests on Trigonometry

Test

1

on lesson 1 – unit 2

Total mark

10

Answer the following questions :

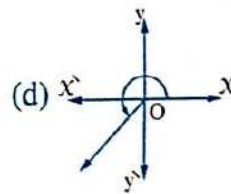
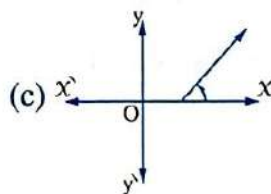
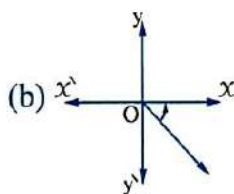
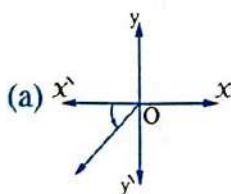
First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

- (1) The angle of measure 50° in the standard position is equivalent to the angle of measure
- (a) 130° (b) 310° (c) 140° (d) 410°
- (2) All the following are measures of angles that lie in the second quadrant except
- (a) -210° (b) 120° (c) -120° (d) 850°
- (3) The angle whose measure is (-750°) lies in the quadrant.
- (a) first (b) second (c) third (d) fourth
- (4) All the following directed angles are not in standard position except



Second question

6 marks

[a] 3 marks

[b] 3 marks

[a] Determine the quadrant in which each of the following angles lie :

(1) -52°

(2) 220°

(3) $1120^\circ 15'$

[b] Find two angles , one of them with positive measure and the other with negative measure having common terminal side for each of the following angles :

(1) -132°

(2) 70°

(3) -730°

Third question

2 marks

* Fig. (1) : 1 mark

* Fig. (2) : 1 mark

Find the measure of the directed angle θ in each of the two figures :

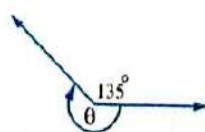


Fig. (1)



Fig. (2)



Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

- (1) The angle whose measure is $\frac{9\pi}{4}$ lies in the quadrant.
 (a) first (b) second (c) third (d) fourth
- (2) The degree measure of a central angle in a circle of radius length 6 cm. and opposite to an arc of length 3π cm. equals
 (a) 30° (b) 60° (c) 90° (d) 120°
- (3) The angle whose measure is -7.3^{rad} is equivalent to the angle whose degree measure is
 (a) $58^\circ 15' 33''$ (b) $301^\circ 44' 27''$ (c) $-233^\circ 15' 33''$ (d) $211^\circ 44' 27''$
- (4) The radian measure of the central angle subtending an arc of length 3 cm. in a circle whose diameter length is 4 cm. equals
 (a) $\left(\frac{2}{3}\right)^{\text{rad}}$ (b) $\left(\frac{3}{2}\right)^{\text{rad}}$ (c) 5^{rad} (d) 6^{rad}

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] Find the length of the arc which is opposite to an inscribed angle of measure 60° , in a circle whose radius length is 10 cm.

[b] ABC is a triangle in which : $m(\angle A) = 70^\circ$, $m(\angle B) = 60^\circ$,
 find in radian measure $m(\angle C)$

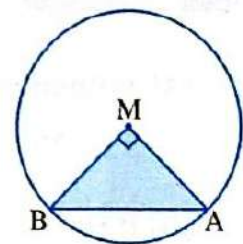
Third question

4 marks

In the opposite figure :

If the area of $\triangle AMB$ equals 8 cm^2 ,

find the length of : \widehat{AB}





Answer the following questions :

First question**2 marks**each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

- (1) The radian measure of the central angle which subtends an arc of length 5 cm. in a circle of diameter length 10 cm. equals
- (a) $\frac{1}{2}^{\text{rad}}$ (b) 1^{rad} (c) 2^{rad} (d) π
- (2) The measure of the smallest positive angle equivalent to the angle whose measure is (-870°) is
- (a) 210° (b) 150° (c) -210° (d) 120°
- (3) If θ is the measure of a directed angle drawn in the standard position where $\sin \theta < 0$, in which quadrant does the terminal side of the angle θ lie ?
- (a) first. (b) first and second.
(c) second and third. (d) third and fourth.
- (4) If $\sec \theta = 2$ where θ is the measure of an acute positive angle, then $\theta = \dots\dots\dots$
- (a) 30° (b) 60° (c) 45° (d) 90°

Second question**4 marks**

[a] 2 marks

[b] 2 marks

[a] Convert the degree measure to the radian measure and the radian measure to the degree measure :

(1) 225° (2) $\frac{8\pi}{5}$

[b] A central angle of measure 60° subtends an arc of length $\frac{7\pi}{3}$ cm. , calculate the radius length of its circle.

Third question**4 marks**

[a] 2 marks

[b] 2 marks

[a] Without using calculator, find the value of :

$$3 \sin 30^\circ \sin^2 60^\circ - \cos 0^\circ \sec 60^\circ + \sin 270^\circ \cos^2 45^\circ$$

[b] If $\sin \theta = \frac{3}{5}$, $\theta \in \left] \frac{\pi}{2}, \pi \right[$, find all trigonometric functions of the angle whose measure is θ

Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) The simplest form of the expression : $\tan (180^\circ + \theta) + \cot (270^\circ - \theta)$ is

- (a) 0 (b) $2 \tan \theta$ (c) $2 \cot \theta$ (d) 2

(2) If $\sin \theta > 0$, $\tan \theta < 0$, then θ lies in the quadrant.

- (a) first (b) second (c) third (d) fourth

(3) If θ is the measure of an acute angle , $\cos (\theta + 25^\circ) = \sin 30^\circ$, then $\theta =$

- (a) 5° (b) 20° (c) 25° (d) 35°

(4) The degree measure of the central angle which subtends an arc of length 3π cm. in a circle of radius length 4 cm. is

- (a) $\frac{3\pi}{4}$ (b) 45° (c) 135° (d) 270°

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] If the terminal side of an angle θ drawn in the standard position intersects the unit circle at the point $(-\frac{3}{5}, -\frac{4}{5})$, find in the simplest form the value of the expression :
 $\cos (180^\circ - \theta) \cot (90^\circ - \theta) + \sin (180^\circ - \theta) \tan (-\theta)$

[b] Find the general solution of the equation :

$\csc (2\theta - 15^\circ) = \sec (\theta - 30^\circ)$, then find all the values of θ where $\theta \in]0^\circ, 90^\circ[$ which satisfy the equation.

Third question

4 marks

[a] 2 marks

[b] 2 marks

[a] Without using calculator , find the value of the expression :

$$\sin (-30^\circ) \sin 240^\circ - \tan 135^\circ \sec 240^\circ$$

[b] If θ is the measure of a central angle in a circle whose radius length is r and subtends an arc of length ℓ where $r = 12$ cm. , $\theta = 1.6^{\text{rad}}$, then find : ℓ



Answer the following questions :

First question**2 marks**each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

- (1) The maximum value of the function $f : f(\theta) = 4 \sin 2\theta$ is
(a) 4 (b) -4 (c) 2 (d) -2
- (2) The angle of measure 620° lies in the quadrant.
(a) first (b) second (c) third (d) fourth
- (3) The radian measure of the angle whose measure is 120° in terms of π is
(a) $\frac{1}{3}\pi$ (b) $\frac{2}{3}\pi$ (c) $\frac{3}{2}\pi$ (d) $\frac{1}{2}\pi$
- (4) If $\sin \theta = \cos 2\theta$ where $\theta \in]0^\circ, 90^\circ[$, then $\sin 3\theta =$
(a) $\frac{1}{2}$ (b) 1 (c) zero (d) $\frac{\sqrt{3}}{2}$

Second question**4 marks**

[a] 2 marks

[b] 2 marks

[a] Find the general solution of the equation : $\tan 4\theta = \cot 2\theta$

[b] Without using calculator, find the value of the expression :

$$\cos(-315^\circ) \sin 405^\circ - \sin 330^\circ \tan 135^\circ$$

Third question**4 marks**

[a] 2.5 marks

[b] 1.5 marks

[a] An angle of measure θ in the standard position, its terminal side intersects the unit circle at the point B $(4a, -3a)$ where $a > 0$

Find the value of each of :

- (1) $\sin(90^\circ + \theta)$ (2) $\csc(-\theta)$

[b] Complete :

The function $f : f(\theta) = \cos \theta$

- (1) Its domain is (2) Its range is
(3) Its period is



Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

- (1) If $2 \cos \theta = -\sqrt{2}$, then the measure of the smallest positive angle satisfying that is
- (a) 45° (b) 135° (c) 225° (d) 315°
- (2) The simplest form of the expression : $\tan (360^\circ - \theta) + \cot (270^\circ - \theta)$ is
- (a) zero (b) 2 (c) $2 \tan \theta$ (d) $2 \cot \theta$
- (3) The degree measure of the central angle which subtends an arc of length 6π cm. in a circle of radius length 9 cm. is
- (a) 30° (b) 60° (c) 120° (d) 150°
- (4) Which of the following angles whose sine and cosine are negative ?
- (a) 50° (b) 150° (c) 210° (d) 300°

Second question

4 marks

[a] 2 marks

[b] 2 marks

[a] Find in degree measure the value of θ which satisfies : $\cos \theta = -0.642$

[b] If the terminal side of a directed angle whose measure is θ in the standard position

intersects the unit circle at the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, find the value of : θ

Third question

4 marks

[a] 2 marks

[b] 2 marks

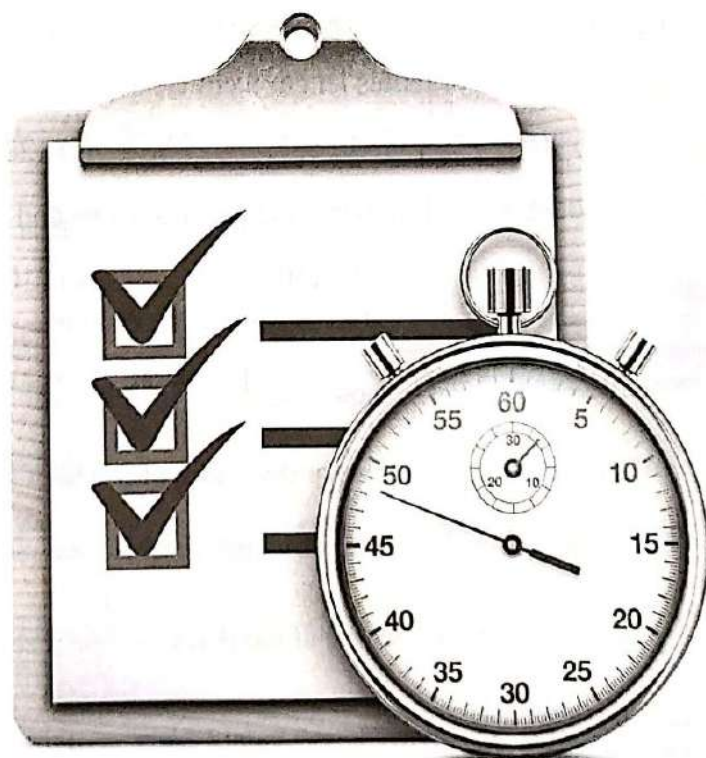
[a] If $25 \sin \theta = 7$ where $\frac{\pi}{2} < \theta < \pi$, find the numerical value of the expression :

$$\tan (\pi + \theta) - \cot \left(\theta - \frac{\pi}{2} \right)$$

[b] Without using calculator, find in the simplest form the value of the expression :

$$\sin (-60^\circ) \cos 390^\circ + \frac{\tan 72^\circ}{\cot 18^\circ}$$

Final Revision on Algebra and Trigonometry



Final Revision on Algebra

Remember The complex numbers

The imaginary number "i"

The imaginary number "i" is defined as the number whose square is -1 i.e. $i^2 = -1$

Notice that

$$\bullet i \times i = i^2 = -1$$

$$\bullet -i \times -i = i^2 = -1$$

$$\bullet \sqrt{-2} = \sqrt{2 i^2} = \sqrt{2} i \quad \text{Similarly :}$$

$$\bullet \sqrt{-5} = \sqrt{5} i$$

$$\bullet \sqrt{-9} = 3 i$$

Integer powers of "i"

To find i^m where m is an integer

We find the remainder of $m \div 4$, if :

The remainder = 0 then $i^m = 1$

The remainder = 1 then $i^m = i$

The remainder = 2 then $i^m = -1$

The remainder = 3 then $i^m = -i$

For example :

$$\bullet i^{12} = 1 \quad \text{"because } 12 \div 4 = 3 \text{ and the remainder is } 0\text{"}$$

$$\bullet i^{63} = -i \quad \text{"because } 63 \div 4 = 15 \text{ and the remainder is } 3\text{"}$$

$$\bullet i^{101} = i \quad \text{"because } 101 \div 4 = 25 \text{ and the remainder is } 1\text{"}$$

$$\bullet i^{26} = -1 \quad \text{"because } 26 \div 4 = 6 \text{ and the remainder is } 2\text{"}$$

$$\bullet i^{12n+3} \text{ "where } n \in \mathbb{Z}\text{"} = -i \quad \text{"because } \frac{12n+3}{4} = 3n \text{ and the remainder is } 3\text{"}$$

Remark

We can express the whole one by using the imaginary number to integer powers from the multiples of the number 4, and this helps in simplifying some imaginary numbers.

$$\text{For example : } \bullet \frac{1}{i^{19}} = \frac{i^{20}}{i^{19}} = i$$

$$\bullet i^{-61} = i^{-61} \times i^{64} = i^3 = -i$$

The complex number

The complex number is the number that can be written in the form : $Z = a + bi$ where a and b are two real numbers, $i^2 = -1$

Examples for complex numbers : $13 - 2i$, $7 + \sqrt{5}i$, -25 , $8i$, $\sqrt{15}$, $5i - 4$

Equality of two complex numbers

Two complex numbers are equal if and only if the two real parts are equal and the two imaginary parts are equal, and vice versa.

If $Z_1 = -5 + xi$, $Z_2 = y + \sqrt{3}i$ and $Z_1 = Z_2$, then $y = -5$, $x = \sqrt{3}$

Adding and subtracting complex numbers

When adding and subtracting two complex numbers, we add or subtract real parts together and add or subtract imaginary parts together.

For example : • $(4 + 5i) + (-2 - 3i) = (4 - 2) + (5 - 3)i = 2 + 2i$

• $(26 - 4i) - (9 - 20i) = (26 - 9) + (-4 + 20)i = 17 + 16i$

Multiplying complex numbers

We use the same properties of multiplying algebraic expressions and multiplying by inspection which we have studied before.

For example : • $2i(1 - 3i) = 2i - 6i^2$ (where $i^2 = -1$) $= 6 + 2i$

• $(3 - 5i)(2 + i) = 6 - 7i - 5i^2$ (where $i^2 = -1$) $= 11 - 7i$

• $(4 - i)^2 = 16 - 8i + i^2$ (where $i^2 = -1$)
 $= 15 - 8i$

• $(5 - 3i)(5 + 3i) = 25 - 9i^2$ (where $i^2 = -1$)
 $= 25 + 9 = 34$

Remember that

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

Remember that

$$(a + b)(a - b) = a^2 - b^2$$

The two conjugate numbers

The two numbers $a + bi$ and $a - bi$ are called conjugate numbers and we notice that the complex number and its conjugate differ only in the sign of their imaginary parts, and their sum is a real number and their product is a real number.

For example :

- The two numbers $3 + 4i$ and $3 - 4i$ are conjugate numbers, while the two numbers $2i - 5$ and $2i + 5$ are not conjugate because the imaginary part in each of them has the same sign.
- The conjugate of the number $4i$ is $-4i$ • The conjugate of the number 6 is 6

Remark

To simplify the fraction whose denominator is a complex number not real, we multiply its two terms by the conjugate of denominator.

For example : $\frac{30 + 45i}{1 - 2i} = \frac{30 + 45i}{1 - 2i} \times \frac{1 + 2i}{1 + 2i} = \frac{30 + 105i + 90i^2}{1 - 4i^2} = \frac{-60 + 105i}{5} = -12 + 21i$

Remember

The quadratic equation in one variable (Determining the type of roots - Finding the solution set)

First**Algebraic method**

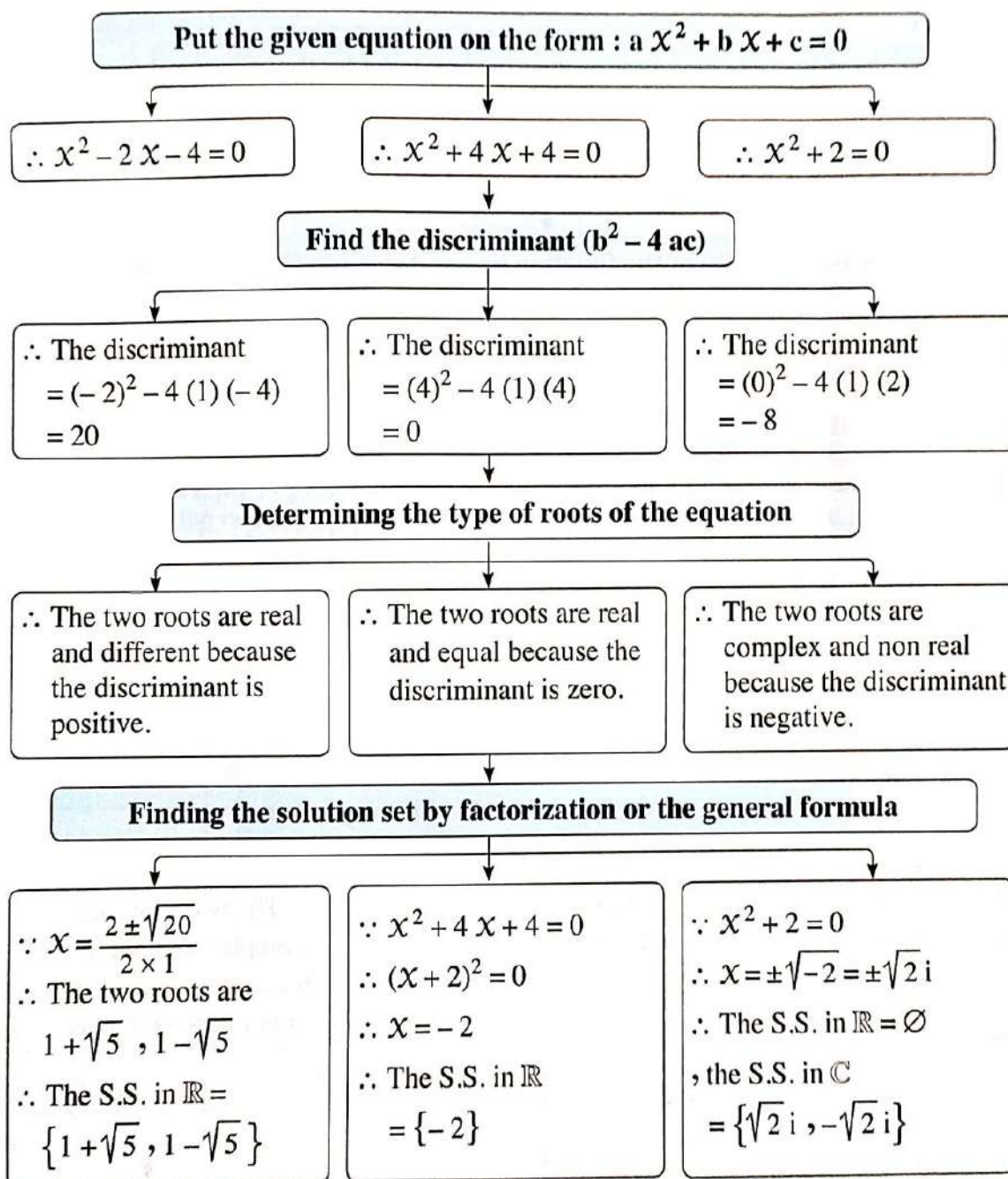
To determine the type of roots of the quadratic equation and find its solution set in \mathbb{R} or in \mathbb{C} for each of the following equations algebraically :

$$\bullet x^2 - 2x - 4 = 0$$

$$\bullet 4x + x^2 + 4 = 0$$

$$\bullet 2 + x^2 = 0$$

We will follow the following steps :



Second Graphic method

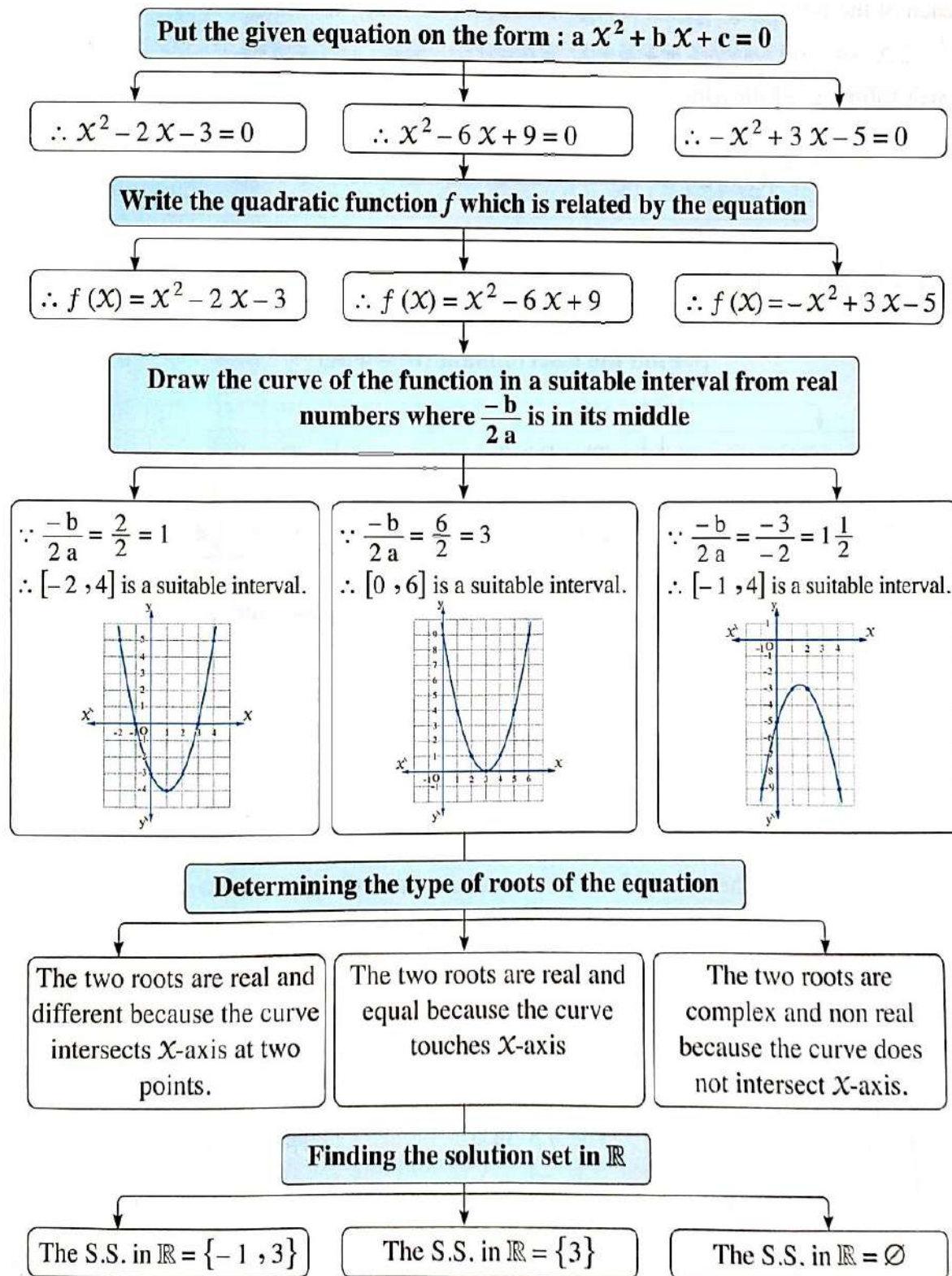
To determine the type of roots of the quadratic equation and find the solution set for each of the following equations graphically :

$$\bullet x^2 - 2x - 3 = 0$$

$$\bullet 9 + x^2 - 6x = 0$$

$$\bullet -x^2 + 3x - 5 = 0$$

We will follow the following steps :



**The relation between the two roots of the equation :
 $aX^2 + bX + c = 0$ and the coefficients of its terms**

The sum of the two roots = $-\frac{b}{a}$

The product of the two roots = $\frac{c}{a}$

For example :

Equation of second degree	The sum of the two roots	The product of the two roots
• $2X^2 + 5X - 4 = 0$	$-\frac{5}{2} = -2.5$	$-\frac{4}{2} = -2$
• $3X^2 - 7X + 3 = 0$	$\frac{7}{3}$	$\frac{3}{3} = 1$ (One of the roots is the multiplicative inverse of the other)
• $5X^2 - 7 = 0$	Zero (One of the roots is the additive inverse of the other)	$-\frac{7}{5}$

Remember Forming the quadratic equation

First Forming the quadratic equation whose two roots are known

We find the sum of the two roots and their product , then the equation will be in the form :

$$X^2 - (\text{the sum of the two roots}) X + \text{the product of the two roots} = 0$$

For example :

If the two roots are	then the sum of the two roots is	the product of the two roots is	Thus , the required equation is
• $3, -4$	-1	-12	$X^2 + X - 12 = 0$
• $\frac{2}{3}, \frac{3}{2}$	$\frac{13}{6}$	1	$X^2 - \frac{13}{6}X + 1 = 0$ i.e. $6X^2 - 13X + 6 = 0$
• $2+i, 2-i$	4	5	$X^2 - 4X + 5 = 0$

Second Forming a quadratic equation from another given quadratic equation**First method**

This method is used if finding the two roots of the given equation is easy.

For example :

If L and M are the two roots of the equation : $x^2 - x - 6 = 0$ where $L > M$

, form the quadratic equation whose roots are : $L - 2$, $M^2 + 1$

1 We find the two roots of the given equation L and M :

$$\because x^2 - x - 6 = 0 \quad \therefore (x - 3)(x + 2) = 0$$

$$\therefore L = 3, M = -2$$

2 We find the two roots of the required equation D and E :

$$\bullet D = L - 2 = 3 - 2 = 1$$

$$\bullet E = M^2 + 1 = (-2)^2 + 1 = 5$$

3 We form the required equation :

$$\therefore x^2 - 6x + 5 = 0$$

Second method

This method is used if we can find " $D + E$ ", " DE " of the required equation in terms of " $L + M$ ", " LM " of the given equation by one of the following identities :

$$\textcircled{1} L^2 + M^2 = (L + M)^2 - 2LM$$

$$\textcircled{2} (L - M)^2 = (L + M)^2 - 4LM$$

$$\textcircled{3} L^3 + M^3 = (L + M) [(L + M)^2 - 3LM]$$

$$\textcircled{4} L^3 - M^3 = (L - M) [(L + M)^2 - LM]$$

$$\textcircled{5} \frac{1}{M} + \frac{1}{L} = \frac{L + M}{LM}$$

$$\textcircled{6} \frac{L}{M} + \frac{M}{L} = \frac{L^2 + M^2}{LM} = \frac{(L + M)^2 - 2LM}{LM}$$

For example :

If L and M are the two roots of the equation : $x^2 - 3x + 1 = 0$

, form the equation whose roots are : $D = \frac{L}{M}$, $E = \frac{M}{L}$

1 We find $L + M$, LM from the given equation :

$$\bullet L + M = \frac{-(-3)}{1} = 3$$

$$\bullet LM = \frac{1}{1} = 1$$

2 We find $D + E$, DE of the required equation in terms of L and M :

$$\bullet D + E = \frac{L}{M} + \frac{M}{L} = \frac{L^2 + M^2}{ML}$$

$$\bullet DE = \frac{L}{M} \times \frac{M}{L} = 1$$

3 We use a suitable identity :

$$\bullet D + E = \frac{L^2 + M^2}{ML} = \frac{(L + M)^2 - 2LM}{ML} = \frac{(3)^2 - 2(1)}{1} = 7$$

4 We form the required equation :

$$\therefore x^2 - (D + E)x + DE = 0$$

$$\text{i.e. } x^2 - 7x + 1 = 0$$

Third method

This method is used only if the relation between D and L is the same relation between E and M

For example :

If L and M are the two roots of the equation : $x^2 - 5x + 2 = 0$

, form the equation whose roots are : $D = L - 3$, $E = M - 3$

1 We find L or M in terms of D or E from the given relation :

$$\therefore D = L - 3$$

$$\therefore L = D + 3$$

2 $\therefore L$ and M are the two roots of the given equation

$\therefore L$ and M satisfy the given equation

$$\therefore (D + 3)^2 - 5(D + 3) + 2 = 0$$

$$\therefore D^2 + 6D + 9 - 5D - 15 + 2 = 0$$

$$\therefore D^2 + D - 4 = 0$$

3 We write the required equation :

$\therefore D$ is one of the roots of the required equation

\therefore The required equation is : $x^2 + x - 4 = 0$

Remember The sign of the function**The sign of the constant function**

The sign of the constant function $f : f(x) = c, c \in \mathbb{R}^*$ is the same sign of c for all values of $x \in \mathbb{R}$

For example :

- The sign of the function $f : f(x) = -7$ is negative for all values of $x \in \mathbb{R}$
- The sign of the function $f : f(x) = 2$ is positive for all values of $x \in \mathbb{R}$

The sign of the first degree function (linear function)

To determine the sign of the linear function $f : f(x) = bx + c, b \neq 0$, we put $f(x) = 0$ $\therefore bx + c = 0$ $\therefore x = \frac{-c}{b}$

Then the sign of the function f :

①

Is the same sign of b at
 $x > \frac{-c}{b}$

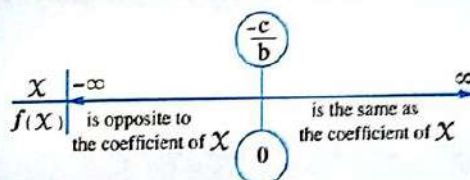
②

Is opposite to the sign of b at
 $x < \frac{-c}{b}$

③

$f(x) = 0$ at
 $x = \frac{-c}{b}$

And we illustrate this on the number line as in the figure :



For example :

If $f : f(x) = -3x + 6$ Put $-3x + 6 = 0$ $\therefore x = 2$

The sign of the function f :

①

Is negative at $x > 2$

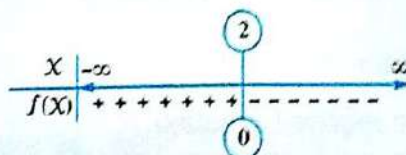
②

Is positive at $x < 2$

③

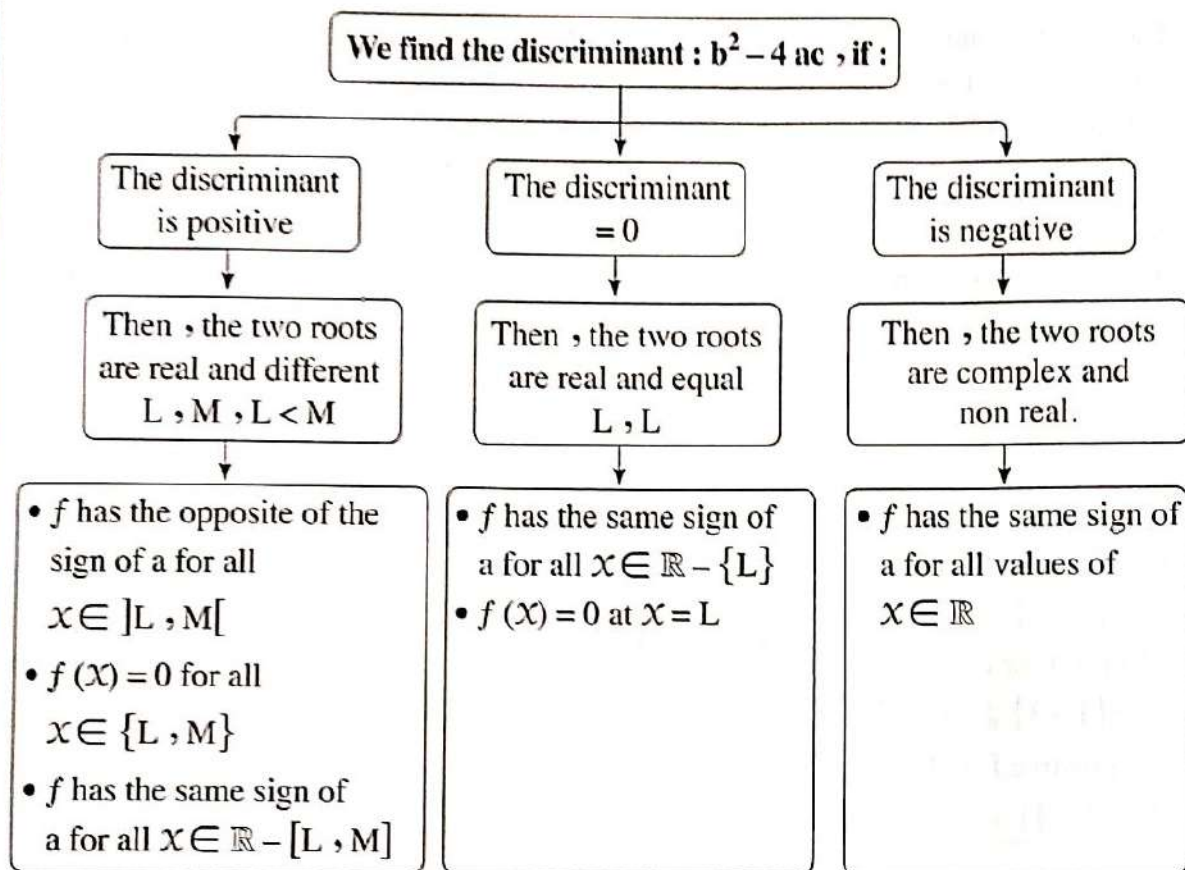
$f(x) = 0$ at $x = 2$

And we illustrate this on the number line as in the figure :



The sign of the second degree function (quadratic function)

To determine the sign of the quadratic function $f : f(x) = ax^2 + bx + c$, $a \neq 0$, we write the quadratic equation : $ax^2 + bx + c = 0$ which is related by the function, then do the following steps :



For example :

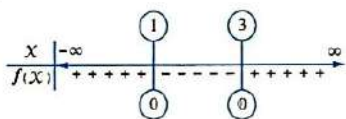
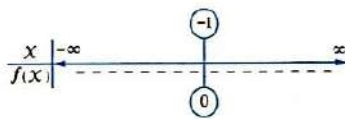
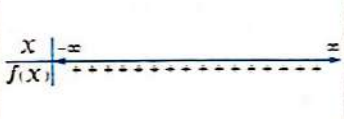
If : • $f : f(x) = x^2 - 4x + 3$

• $f : f(x) = -x^2 - 2x - 1$

• $f : f(x) = 2x^2 - 3x + 5$

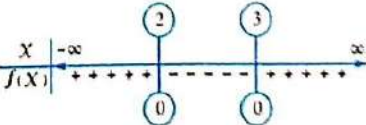

, then we can determine the sign of each of the previous functions as the following :

We write the quadratic equations which are related by the previous functions and complete the steps as follows :

$x^2 - 4x + 3 = 0$	$x^2 + 2x + 1 = 0$	$2x^2 - 3x + 5 = 0$
\therefore The discriminant $= (-4)^2 - 4 \times 1 \times 3$ $= 4$ (positive)	\therefore The discriminant $= (2)^2 - 4 \times 1 \times 1 = 0$	\therefore The discriminant $= (-3)^2 - 4 \times 2 \times 5$ $= -31$ (negative)
\therefore The two roots are real and different and they are 3 and 1	\therefore The two roots are real and equal and each of them equals -1	\therefore The two roots are complex and non real
 <ul style="list-style-type: none"> f is negative for all $x \in]1, 3[$ $f(x) = 0$ for all $x \in \{1, 3\}$ f is positive for all $x \in \mathbb{R} - [1, 3]$ 	 <ul style="list-style-type: none"> f is negative for all $x \in \mathbb{R} - \{-1\}$ $f(x) = 0$ at $x = -1$ 	 <ul style="list-style-type: none"> f is positive for all values of $x \in \mathbb{R}$

Remember the solving of the quadratic inequalities in \mathbb{R}

To find the solution set of the inequality : $x^2 - 5x + 6 > 0$ in \mathbb{R} :

<p>① We write the quadratic function related by the inequality.</p> <p>$f : f(x) = x^2 - 5x + 6$</p>	<p>② We study the sign of the quadratic function which we wrote.</p> <p> \therefore The discriminant $= (-5)^2 - 4 \times 1 \times 6$ $= 1$ (positive) \therefore The two roots are real and different $\therefore (x-2)(x-3) = 0$ $\therefore x = 2$ or $x = 3$ </p> 	<p>③ We determine the intervals which satisfy the inequality.</p> <p>The solution set of the inequality : $x^2 - 5x + 6 > 0$ is $\mathbb{R} - [2, 3]$</p> 
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Final Revision on Trigonometry

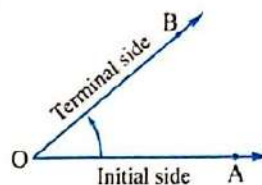
Remember The directed angle

Definition of the directed angle

The directed angle is an ordered pair of two rays called the sides of the angle with a common starting point called the vertex.

For example :

The ordered pair $(\overrightarrow{OA}, \overrightarrow{OB})$ represents the directed angle $\angle AOB$ whose initial side is \overrightarrow{OA} and terminal side is \overrightarrow{OB}



Positive and negative measures of a directed angle

If the positive measure of the directed angle $= \theta$
then the negative measure of the same directed angle $= \theta - 360^\circ$

For example :

The negative measure of the directed angle of measure $210^\circ = 210^\circ - 360^\circ = -150^\circ$

If the negative measure of the directed angle $= -\theta$
then the positive measure of the same directed angle $= -\theta + 360^\circ$

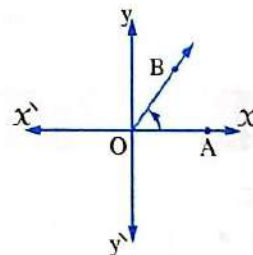
For example :

The positive measure of the directed angle of measure $(-120^\circ) = -120^\circ + 360^\circ = 240^\circ$

The standard position of the directed angle

A directed angle is in the standard position if the following two conditions are satisfied :

- 1 Its initial side lies on the positive direction of the x -axis.
- 2 Its vertex is the origin point of an orthogonal coordinate plane.

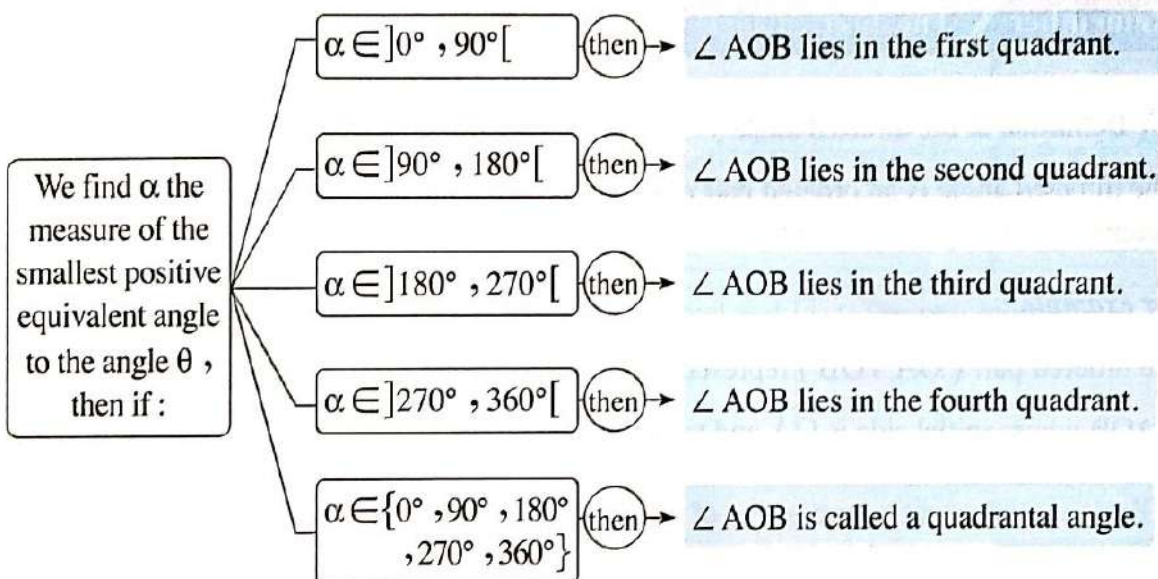


Equivalent angles

Several directed angles in the standard position are said to be equivalent when they have one common terminal side.

And we get equivalent angles to the angle whose measure is θ by adding $n \cdot 360^\circ$ to it or subtracting $n \cdot 360^\circ$ from it where n is an integer.

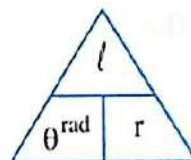
Determining the quadrant in which the terminal side of the directed angle $\angle AOB$ whose measure is θ in the standard position lies :



Radian measure and degree measure of an angle

- The radian measure of a central angle in a circle = $\frac{\text{Length of the arc which the central angle subtends}}{\text{Length of the radius of this circle}}$

i.e. $\theta^{\text{rad}} = \frac{l}{r}$ and from it $l = \theta^{\text{rad}} r$, $r = \frac{l}{\theta^{\text{rad}}}$



- The relation between the radian measure and the degree measure :

$\frac{x^\circ}{180^\circ} = \frac{\theta^{\text{rad}}}{\pi}$ and from it $\theta^{\text{rad}} = x^\circ \times \frac{\pi}{180^\circ}$, $x^\circ = \theta^{\text{rad}} \times \frac{180^\circ}{\pi}$

Notice that

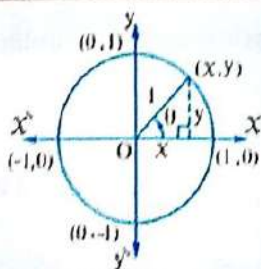
π in radians is equivalent to 180° in degrees.

Remember The trigonometric functions of an acute angle and their reciprocals

• $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = y$

• $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = x$

• $\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$



$x^2 + y^2 = 1$

• $\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{y}$

• $\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{1}{x}$

• $\cot \theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$

Notice that

- $x \in [-1, 1]$ and from it $\cos \theta \in [-1, 1]$
- $y \in [-1, 1]$ and from it $\sin \theta \in [-1, 1]$
- The equivalent angles have the same trigonometric functions.

Remember The signs of trigonometric functions

Quadrant	The interval that θ belongs to	sign of \cos, \sec	sign of \sin, \csc	sign of \tan, \cot	
First	$]0, \frac{\pi}{2}[$	+	+	+	
Second	$] \frac{\pi}{2}, \pi [$	-	+	-	
Third	$] \pi, \frac{3\pi}{2} [$	-	-	+	
Fourth	$] \frac{3\pi}{2}, 2\pi [$	+	-	-	

Notice that

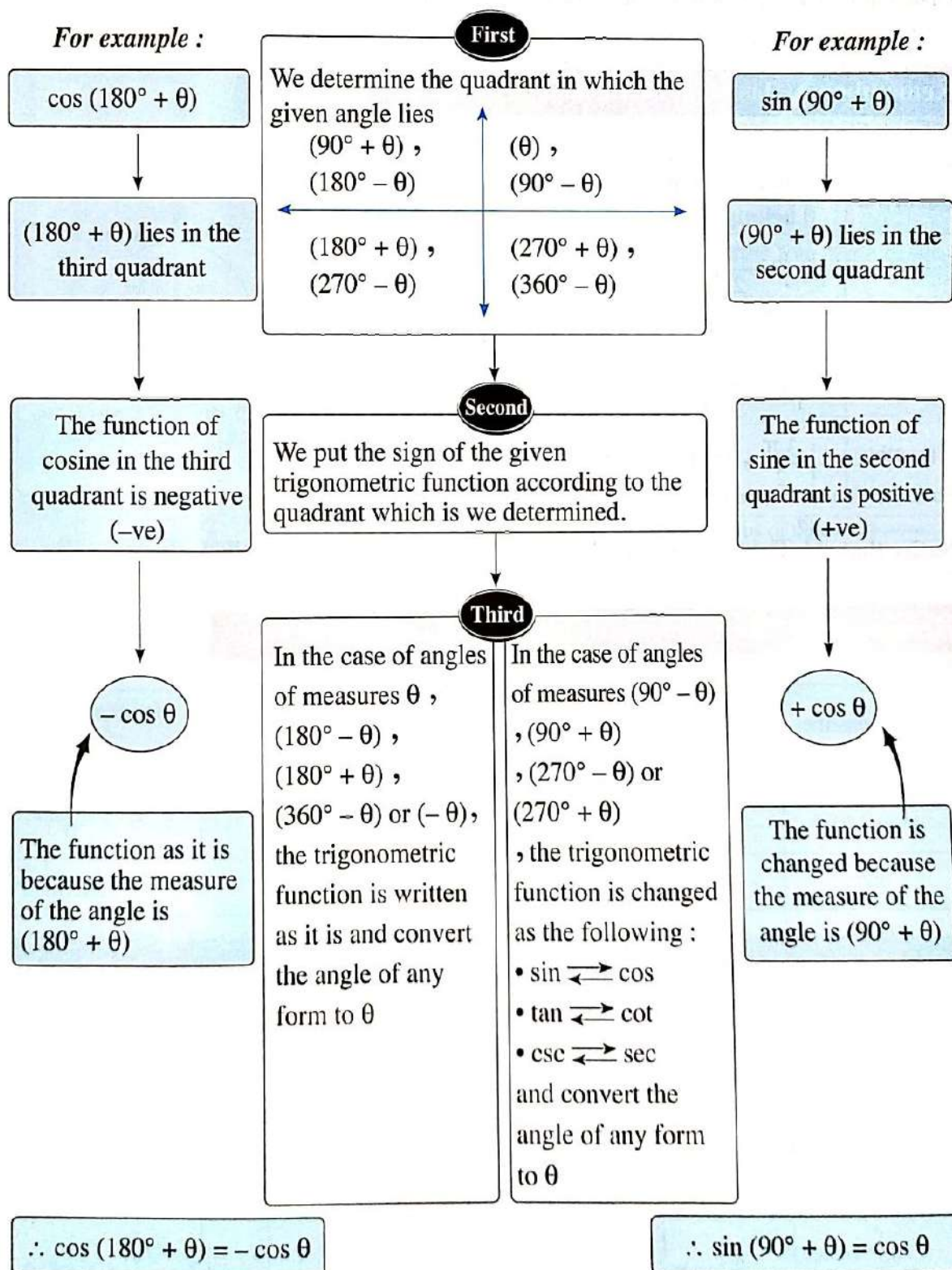
The trigonometric functions of the equivalent angles have the same sign.

Remember The trigonometric functions of some special angles

The measure of θ	The point of the intersection of the terminal side with the unit circle	The values of the trigonometric functions		
		$\sin \theta$	$\cos \theta$	$\tan \theta$
0° or 360°	$(1, 0)$	0	1	0
90°	$(0, 1)$	1	0	undefined
180°	$(-1, 0)$	0	-1	0
270°	$(0, -1)$	-1	0	undefined
30°	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
60°	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
45°	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1

Remember The relation between the trigonometric functions of two related angles

To know how to find the relations between the trigonometric functions of two related angles, we will follow the following steps :



For example :

Without using calculator, we can find :

$$\begin{aligned}
 & \cos(-150^\circ) \sin 600^\circ + \cos \frac{2\pi}{3} \sin 330^\circ - \sec\left(-\frac{5\pi}{4}\right) \tan 900^\circ \\
 &= \cos(210^\circ) \sin(360^\circ + 240^\circ) + \cos 120^\circ \sin(360^\circ - 30^\circ) - \sec 225^\circ \tan(180^\circ + 2 \times 360^\circ) \\
 &= \cos(180^\circ + 30^\circ) \sin(180^\circ + 60^\circ) + \cos(180^\circ - 60^\circ) \sin(360^\circ - 30^\circ) - \sec(180^\circ + 45^\circ) \tan 180^\circ
 \end{aligned}$$

↓
Third quadrant

↓
Third quadrant

↓
Second quadrant

↓
Fourth quadrant

↓
Third quadrant

↓
Quadrantal angle

$$= (-\cos 30^\circ)(-\sin 60^\circ) + (-\cos 60^\circ)(-\sin 30^\circ) - (-\sec 45^\circ) \times 0$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} - 0 = \frac{3}{4} + \frac{1}{4} = 1$$

Remark

If α and β are the measures of two complementary angles (*i.e.* Their sum is 90°), then $\sin \alpha = \cos \beta$, $\tan \alpha = \cot \beta$, $\sec \alpha = \csc \beta$, ...

For example :

20° and 70° are measures of two complementary angles.

$\therefore \sin 20^\circ = \cos 70^\circ$, $\tan 70^\circ = \cot 20^\circ$, ...

Remember

The general solution to solve the equations in the form $\sin \alpha = \cos \beta$ or $\csc \alpha = \sec \beta$ or $\tan \alpha = \cot \beta$

1 If $\sin \alpha = \cos \beta$

, then $\alpha \pm \beta = 90^\circ + 360^\circ n$

i.e. $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

i.e. The measure of angle of sine \pm the measure of angle of cosine = $90^\circ + 360^\circ n$

2 If $\csc \alpha = \sec \beta$

, then $\alpha \pm \beta = 90^\circ + 360^\circ n$

i.e. $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

, $\alpha \neq n\pi$, $\beta \neq (2n+1)\frac{\pi}{2}$

3 If $\tan \alpha = \cot \beta$

, then $\alpha + \beta = 90^\circ + 180^\circ n$

i.e. $\alpha + \beta = \frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$

, $\alpha \neq (2n+1)\frac{\pi}{2}$, $\beta \neq n\pi$

and the following example expresses the previous :

• If $\sin 4\theta = \cos 2\theta$, $\theta \in]0, \frac{\pi}{2}[$

$$\therefore 4\theta \pm 2\theta = \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z}$$

Or

$$\therefore 2\theta = \frac{\pi}{2} + 2\pi n$$

$$\therefore \theta = \frac{\pi}{4} + \pi n$$

• At $n = 0$

$$\therefore \theta = \frac{\pi}{4} = 45^\circ$$

• At $n = 1$

$$\therefore \theta = \frac{\pi}{4} + \pi$$

(refused)

$$6\theta = \frac{\pi}{2} + 2\pi n$$

$$\theta = \frac{\pi}{12} + \frac{\pi}{3}n$$

• At $n = 0$

$$\therefore \theta = \frac{\pi}{12} = 15^\circ$$

• At $n = 1$

$$\therefore \theta = \frac{\pi}{12} + \frac{\pi}{3} = 75^\circ$$

• At $n = 2$

$$\therefore \theta = \frac{\pi}{12} + \frac{2\pi}{3}$$

(refused)

$$\therefore \theta = 15^\circ, 45^\circ \text{ or } 75^\circ$$

• If $\tan 3\theta = \cot 2\theta$, $\theta \in]0, \frac{\pi}{2}[$

$$\therefore 3\theta + 2\theta = \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$$

↓

$$\therefore 5\theta = \frac{\pi}{2} + \pi n$$

$$\therefore \theta = \frac{\pi}{10} + \frac{\pi}{5}n$$

• At $n = 0$

$$\therefore \theta = \frac{\pi}{10} = 18^\circ$$

• At $n = 1$

$$\therefore \theta = \frac{\pi}{10} + \frac{\pi}{5} = \frac{3\pi}{10} = 54^\circ$$

• At $n = 2$

$$\therefore \theta = \frac{\pi}{10} + \frac{2\pi}{5} = \frac{1}{2}\pi$$

(refused)

$$\therefore \theta = 18^\circ \text{ or } 54^\circ$$

Remember

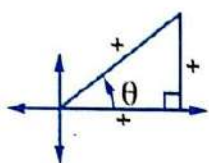
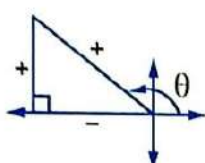
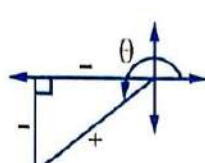
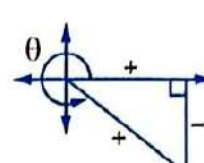
How to find the measure of an angle (θ) given the value of one of its trigonometric ratios (a)

Steps	Examples	$\sin \theta = -\frac{1}{2}$	$\cos \theta = \frac{1}{\sqrt{2}}$	$\tan \theta = -\sqrt{3}$
1	We determine the quadrant in which θ lies according to the sign of a	The sine function is negative. $\therefore \theta$ lies in the third or the fourth quadrant.	The cosine function is positive. $\therefore \theta$ lies in the first or the fourth quadrant.	The tangent function is negative. $\therefore \theta$ lies in the second or the fourth quadrant
2	We find the measure of the acute angle α whose trigonometric function = $ a $	$\sin \alpha = \left -\frac{1}{2} \right = \frac{1}{2}$ $\therefore \alpha = 30^\circ$	$\cos \alpha = \left \frac{1}{\sqrt{2}} \right = \frac{1}{\sqrt{2}}$ $\therefore \alpha = 45^\circ$	$\tan \alpha = \left -\sqrt{3} \right = \sqrt{3}$ $\therefore \alpha = 60^\circ$
3	We put the angle θ in the quadrant that we determined at the first step by using one of the relations : $180^\circ - \alpha$, $180^\circ + \alpha$ or $360^\circ - \alpha$.	$\therefore \theta$ lies in the third quadrant. $\therefore \theta = 180^\circ + \alpha$ $= 180^\circ + 30^\circ$ $= 210^\circ$ or θ lies in the fourth quadrant. $\therefore \theta = 360^\circ - \alpha$ $= 360^\circ - 30^\circ$ $= 330^\circ$	$\therefore \theta$ lies in the first quadrant. $\therefore \theta = \alpha = 45^\circ$ or θ lies in the fourth quadrant $\therefore \theta = 360^\circ - \alpha$ $= 360^\circ - 45^\circ$ $= 315^\circ$	$\therefore \theta$ lies in the second quadrant. $\therefore \theta = 180^\circ - \alpha$ $= 180^\circ - 60^\circ$ $= 120^\circ$ or θ lies in the fourth quadrant $\therefore \theta = 360^\circ - \alpha$ $= 360^\circ - 60^\circ$ $= 300^\circ$

Remember

How to find all the trigonometric functions of an angle given the value of one of its trigonometric functions

We can find the values of the trigonometric functions of an angle directly if we draw the angle in its standard position and we draw the right-angled triangle that represents it by using the value of the given trigonometric function concerning the signs according to the quadrant in which the angle lies as follows :

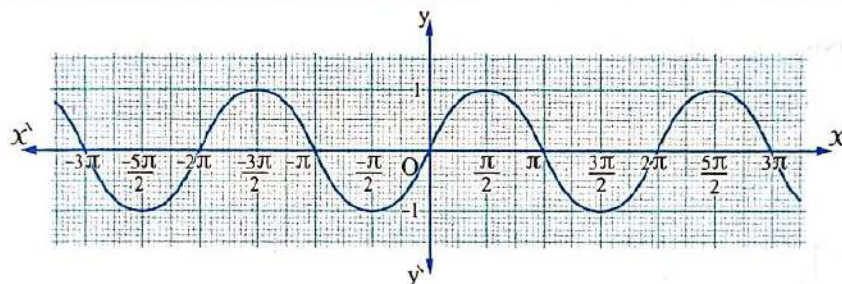
In the 1st quadrantIn the 2nd quadrantIn the 3rd quadrantIn the 4th quadrant

For example :

$\sin \theta = \frac{-8}{17}$ where $270^\circ < \theta < 360^\circ$	$\cos \alpha = \frac{-3}{5}$ where α is the smallest positive angle.	$\tan \beta = \frac{5}{12}$ where β is the greatest positive angle, $0^\circ < \beta < 360^\circ$
$\because 270^\circ < \theta < 360^\circ$ $\therefore \theta$ lies in the fourth quadrant.	$\because \cos \alpha$ is negative $\therefore \alpha$ lies in the second or the third quadrant $\because \alpha$ is the smallest positive angle. $\therefore \alpha$ lies in the second quadrant.	$\because \tan \beta$ is positive $\therefore \beta$ lies in the first or the third quadrant $\because \beta$ is the greatest positive angle. $\therefore \beta$ lies in the third quadrant
$\therefore \cos \theta = \frac{15}{17}$ $\therefore \tan \theta = \frac{-8}{15}, \dots$	$\therefore \sin \alpha = \frac{4}{5}$ $\therefore \tan \alpha = \frac{-4}{3}, \dots$	$\therefore \sin \beta = \frac{-5}{13}$ $\therefore \cos \beta = \frac{-12}{13}, \dots$

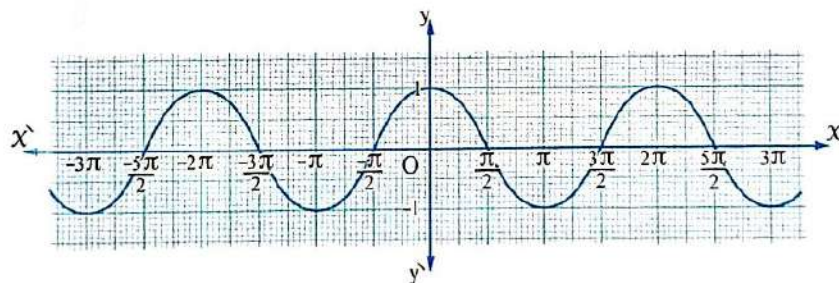
Remember The properties of the sine function and the cosine function

Properties of the sine function $f : f(\theta) = \sin \theta$



- 1 The domain of the sine function is $]-\infty, \infty[$
- 2 • The maximum value of the function is 1 and it happens when $\theta = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$
 • The minimum value of the function is -1 and it happens when $\theta = \frac{3\pi}{2} + 2n\pi, n \in \mathbb{Z}$
- 3 The range of the function is $[-1, 1]$
- 4 The function is periodic and its period is 2π (360°)

Properties of the cosine function $f : f(\theta) = \cos \theta$



- 1 The domain of the cosine function is $]-\infty, \infty[$
- 2 • The maximum value of the function is 1 and it happens when $\theta = \pm 2n\pi, n \in \mathbb{Z}$
 • The minimum value of the function is -1 and it happens when $\theta = \pi \pm 2\pi n, n \in \mathbb{Z}$
- 3 The range of the function is $[-1, 1]$
- 4 The function is periodic and its period is 2π (360°)

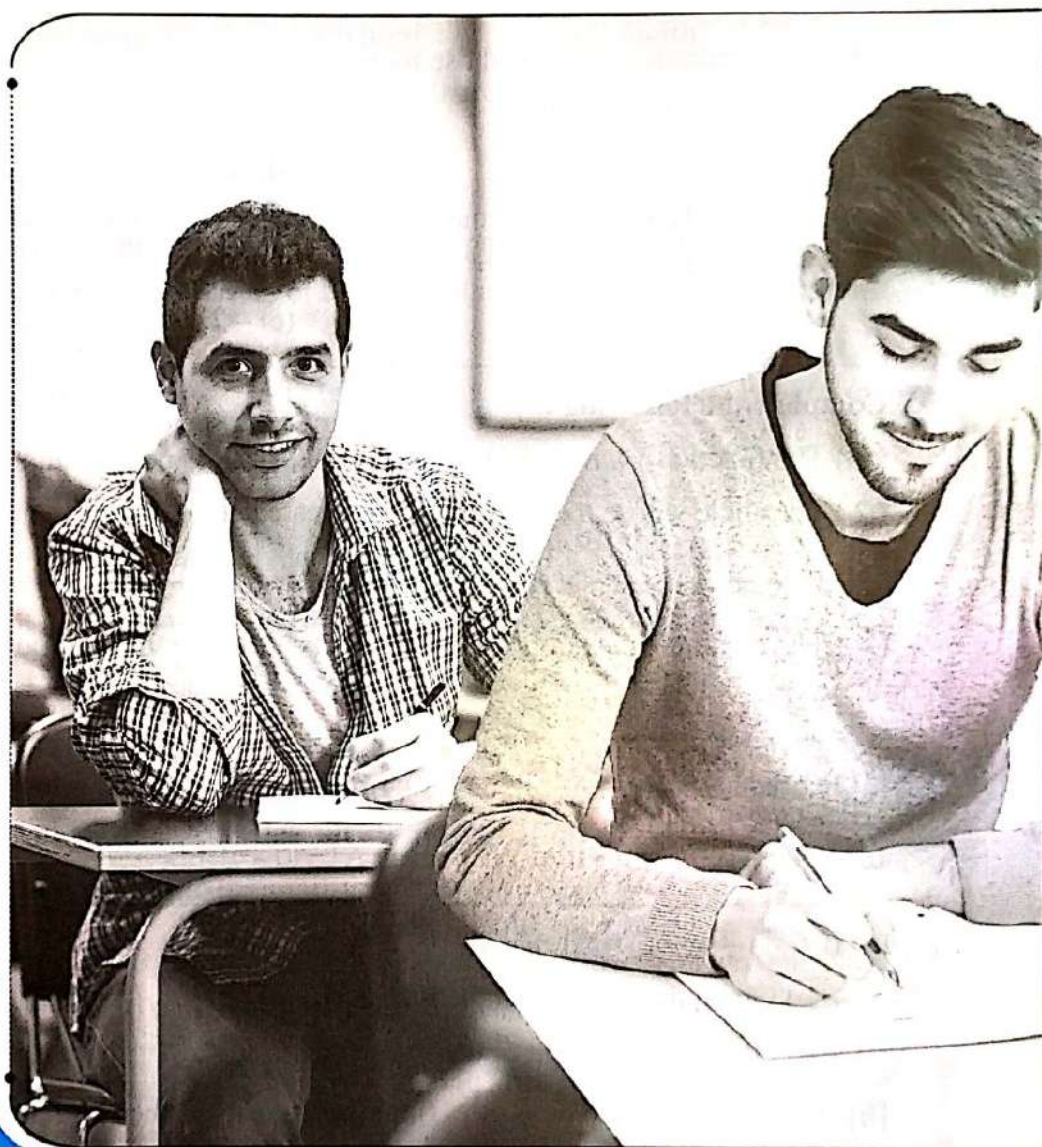
Remark

Each of the two functions $f : f(\theta) = a \sin b\theta$, $f : f(\theta) = a \cos b\theta$ is periodic, its period is $\frac{2\pi}{|b|}$ and its range is $[-a, a]$ where a is positive.

For example : • $f : f(\theta) = 5 \sin \theta$ its period is 2π and its range is $[-5, 5]$

• $f : f(\theta) = 3 \cos 7\theta$ its period is $\frac{2\pi}{7}$ and its range is $[-3, 3]$

Final Examinations of Algebra and Trigonometry



School book examinations

Model 1

1 Choose the correct answer from the given ones :

(1) If L and M are the two roots of the equation : $X^2 - 7X + 3 = 0$, then $L^2 + M^2 = \dots\dots\dots$

- (a) 7 (b) 3 (c) 43 (d) 79

(2) If $\sin \theta = -1$ and $\cos \theta = \text{zero}$, then $\theta = \dots\dots\dots$

- (a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

(3) The quadratic equation whose roots are $2 - 3i$, $2 + 3i$ is $\dots\dots\dots$

- (a) $X^2 + 4X + 13 = 0$ (b) $X^2 - 4X + 13 = 0$
(c) $X^2 + 4X - 13 = 0$ (d) $X^2 - 4X - 13 = 0$

(4) If one of the two roots of the equation : $X^2 - (m + 2)X + 3 = 0$ is the additive inverse of the other root, then $m = \dots\dots\dots$

- (a) 3 (b) 2 (c) -2 (d) -3

2 Complete the following :

(1) The function f where $f(X) = -(X - 1)(X + 2)$ is positive in the interval $\dots\dots\dots$

(2) The angle whose measure is 930° is located at the $\dots\dots\dots$ quadrant.

(3) If $\cos \theta = \frac{1}{2}$ and $\sin \theta = -\frac{\sqrt{3}}{2}$, then $\theta = \dots\dots\dots^\circ$

(4) The quadratic equation whose two roots are twice the two roots of the equation : $2X^2 - 8X + 5 = 0$ is $\dots\dots\dots$

3 [a] Put the number $\frac{2-3i}{3+2i}$ in the form of a complex number where $i^2 = -1$

[b] If $4 \sin A - 3 = 0$, find : A, where $A \in]0, \frac{\pi}{2}[$

4 [a] If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(X) = -X^2 + 8X - 15$

(1) Graph the function in the interval $[1, 7]$

(2) Determine the sign of the function.

[b] If $X = 3 + 2i$ and $y = \frac{4-2i}{1-i}$, then find : $X + y$ in the form of a complex number.

5 [a] Find in \mathbb{R} the solution set of the inequality : $X^2 + 3X - 4 \leq 0$

[b] If $\tan B = \frac{3}{4}$, where $180^\circ < B < 270^\circ$, then find the value of :

$$\cos(360^\circ - B) - \cos(90^\circ - B)$$

Model 2

1 Complete the following :

- (1) The simplest form of the imaginary number i^{43} is
- (2) If the two roots of the equation : $X^2 - 6X + L = 0$ are real and equal , then $L =$
- (3) If $0^\circ < \theta < 90^\circ$ and $\sin 2\theta = \cos 3\theta$, then $\theta =$
- (4) The range of the function f where $f(\theta) = \frac{3}{2} \sin \theta$ is

2 Choose the correct answer :

- (1) The equation : $X^2(X-1)(X+1) = 0$ is a degree equation.
 (a) first (b) second (c) third (d) fourth
- (2) If the two roots of the equation : $X^2 + 3X - m = 0$ are real different , then $m =$
 (a) -2 (b) -3 (c) -4 (d) -5
- (3) If the sum of measures of the angles of a regular polygon equals $180^\circ(n-2)$ where n is the number of sides , then the measure of the angle of a regular octagon by the radian measure equals
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{2\pi}{3}$
- (4) If $2 \cos \theta = -\sqrt{3}$ and $\pi < \theta < \frac{3\pi}{2}$, then $\theta =$
 (a) $\frac{\pi}{3}$ (b) $\frac{6\pi}{7}$ (c) $\frac{4\pi}{3}$ (d) $\frac{7\pi}{6}$

3 [a] Find the value of k which makes one root of the two roots of the equation :

$4kX^2 + 7X + k^2 + 4 = 0$ be the multiplicative inverse of the other root.

[b] If $\sin \theta = \sin 750^\circ \cos 300^\circ + \sin (-60^\circ) \cot 120^\circ$ where $0^\circ < \theta < 360^\circ$, find : θ

4 [a] (1) Find the two values of a , b which satisfy the equation : $12 + 3ai = 4b - 27i$

(2) Find the solution set of the inequality : $X(X+1) - 2 \leq 0$ in \mathbb{R}

[b] A central angle of measure θ is inscribed in a circle of radius length 18 cm. and subtends an arc of length 26 cm. Find θ in degree measure.

5 [a] If the sum of the consecutive integers $(1 + 2 + 3 + \dots + n)$, where n is the number of integers is given by the relation $S = \frac{n}{2}(1+n)$, how many consecutive integers starting from number 1 to be summed 210 are there ?

[b] If $\sin X = \frac{4}{5}$ where $90^\circ < X < 180^\circ$
 , find : $\sin(180^\circ - X) + \tan(360^\circ - X) + 2 \sin(270^\circ - X)$

Some schools examinations

1

Cairo Governorate

Hel. Educ. Administration
St. Joseph's School

Answer the following questions :

1 Choose the correct answer :

(1) 30° to radian measure =

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$

(d) $\frac{1}{2}$

(2) If $\frac{\tan \theta}{\cot 2\theta} = 1$, $0^\circ < \theta < 90^\circ$, then $\theta = \dots^\circ$

(a) 30

(b) 60

(c) 90

(d) 45

(3) $a^i \times a^{i^2} \times a^{i^3} \times a^{i^4} = \dots$

(a) a

(b) 1

(c) 0

(d) a^2

(4) $\frac{\left(x \sin \frac{\pi}{4}\right)^2 - y^2 \cos^2 \frac{\pi}{4}}{x^2 - y^2} = \cos \dots$

(a) $x + y$

(b) $\frac{\pi}{4}$

(c) 60°

(d) $\frac{1}{2}$

2 Complete :

(1) If the equation $aX^2 + bX + c = 0$ has two real equal rational roots, then $b^2 - 4ac \dots$ (2) If $i^3 - 3i$, $1 + 4i$ are the two roots of the equation $X^2 - (k-1)X + b = 0$, then $k = \dots$ (3) If θ is an angle in the standard position and its terminal side passes through the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, then $\cos \theta = \dots$ and $\tan \theta = \dots$ (4) If L , M are the roots of the equation $X^2 - 3X + 2 = 0$, then $L^2 + 2LM + M^2 = \dots$

3 [a] Without using calculator find the value of :

$$3 \sin 30^\circ \sin 60^\circ - \cos 0^\circ \sec 60^\circ + \sin 270^\circ \cos^2 45^\circ$$

[b] Find in \mathbb{R} the solution set of the inequality : $(X+2)(X-3) \leq 0$ 4 [a] If $\sin(180^\circ - X) = \cos 60^\circ \sin 270^\circ + \cot 120^\circ \sin(-60^\circ)$, where $X \in]0^\circ, 360^\circ[$ Find : $m(\angle X)$

- [b] Graph the curve of the function f , where $f(x) = x^2 - 1$, from the graph determine the sign of the function f

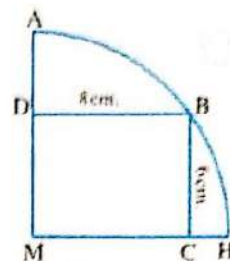
5 [a] Prove that : $\left[\left(\sqrt{-1} \right)^{8n+3} - \left(\sqrt{-1} \right)^{2(2n+1)} \right]^4 = -4$

[b] In the opposite figure :

Quarter of circle M, MCBD is a rectangle inside it

, $BD = 8$ cm. , $BC = 6$ cm.

Find the length of the arc : \widehat{ABH}



2

Cairo Governorate

Directing Mathematics
Maadi Kawmia School

Answer the following questions :

1 Choose the correct answer :

- (1) The degree measure of the central angle in a circle of radius length 12 cm. and subtends an arc of length 4π cm. equals

(a) 60° (b) 120° (c) 30° (d) 90°

- (2) If one of the roots of the equation $(1-a)x^2 + 2x = -5$ is the multiplicative inverse of the other root, then $a = \dots\dots\dots$

(a) 4 (b) 2 (c) -4 (d) -2

- (3) If θ is a positive acute angle where $\sqrt{3} \csc \theta = 2$, then $\tan \theta = \dots\dots\dots$

(a) $\frac{1}{2}$ (b) 1 (c) 0 (d) $\sqrt{3}$

- (4) The solution set of the inequality $4x - 16 - x^2 < 0$

(a) $[3, 9]$ (b) $]3, 9[$ (c) \mathbb{R} (d) \emptyset

2 Complete the following :

- (1) If $\csc(\theta + 20^\circ) = \sec(3\theta + 30^\circ)$ where $0^\circ < \theta < 90^\circ$, then $\cos 6\theta = \dots\dots\dots$

- (2) If $\tan \theta = \sqrt{3}$ and $90^\circ < \theta < 360^\circ$, then $\theta = \dots\dots\dots^\circ$

- (3) The range of the function $f(\theta) = 2 \cos \theta$ is

- (4) If n is an integer, then the simplest form of the imaginary $i^{4n+2018}$ is

3 [a] If $x \in \mathbb{R}$, $y \in \mathbb{R}$ find the values of x, y which satisfy the equation :
 $2x - y + xi - 3yi = (2 + i)^2$

[b] If $\angle AOB$ in the standard position, its terminal side intersects the unit circle at the point $B\left(\frac{-4}{5}, y\right)$ where $y < 0$ and $m(\angle AOB) = \theta$, then find :

(1) The value of y (2) $\cos(90^\circ - \theta)$

(3) $\sin(180^\circ - \theta)$

- 4 [a] If L, M are the two roots of the equation : $4x^2 + 3x = 2$

Find the equation whose two roots are : $L - 2, M - 2$

- [b] If $\sin \theta = \frac{3}{5}$, where $\frac{\pi}{2} < \theta < \pi$ Find the value of : $\cot \theta + \cos \left(\frac{\pi}{2} + \theta \right) - \sin (2\pi - \theta)$

- 5 [a] Determine the sign of the function f where $f(x) = x^2 - 7x - 8$

and from this find in \mathbb{R} the S.S. of the inequality : $f(x) \leq 0$

- [b] Without using calculator find the value of :

$$\frac{\sin 15^\circ}{\sin 165^\circ} + \cos 420^\circ + \tan^2 65^\circ - \tan 245^\circ \tan 65^\circ$$

3

Cairo Governorate

Al-Khalifa and Al-Mokattam Direeforate
Al-Waha Language Schools



Answer the following questions :

- 1 Choose the correct answer :

(1) The simplest form of the imaginary number i^{19} is

- (a) i (b) $-i$ (c) 1 (d) -1

(2) If $\cos \theta = \frac{1}{2}$ and $\sin \theta = \frac{-\sqrt{3}}{2}$, then $\theta = \dots^\circ$

- (a) 60 (b) 240 (c) 300 (d) 120

(3) The function f where $f(x) = 3 - 2x$ is negative when $x \in \dots$

- (a) $]1.5, \infty[$ (b) $\{1.5\}$ (c) $]-\infty, 1.5[$ (d) $\mathbb{R} - \{1.5\}$

(4) The angle whose measure is (-930°) lies in the quadrant.

- (a) first (b) second (c) third (d) fourth

- 2 Complete :

(1) If the two roots of the equation $5x^2 + 8x + k = 0$ are multiplicative inverses of each other, then $k = \dots$

(2) The range of the function f where $f(x) = \sin x$ is

(3) $\cot (270^\circ - \theta) + \tan (-\theta) = \dots$

(4) The quadratic equation whose roots are i and $-i$ is

- 3 [a] Determine the sign of the function $f : f(x) = x^2 - 8x + 15$

, then deduce in \mathbb{R} the S.S. of the inequality : $x^2 - 8x + 15 \leq 0$

[b] The measure of a central angle is 72° in a circle of diameter 12 cm.

Find the length of the arc opposite to this angle to the nearest two decimals.

4 [a] Put the number $\frac{26}{3-2i}$ in the form of $a + bi$ (Show your steps)

[b] Determine the type of the roots of the equation : $-x^2 + 5x - 7 = 0$ (State reason)

[c] Find a value for θ that satisfies the equation : $\tan(\theta + 20^\circ) = \cot(3\theta + 30^\circ)$

5 [a] If L and M are the roots of the equation : $x^2 - 6x + 13 = 0$

, form the equation whose roots are : $\frac{L}{M}$ and $\frac{M}{L}$

[b] If $\tan A = \frac{3}{4}$ where $\pi < A < \frac{3\pi}{2}$

Find without using calculator : $\sin(180^\circ - A) - \sin(90^\circ + A)$

4

Giza Governorate

Agouza Educational Directorate
The supervision of Mathematics



Answer the following questions :

1 Choose the correct answer :

(1) The simplest form of the imaginary number $i^{15} = \dots\dots\dots$

- (a) i (b) $-i$ (c) 1 (d) -1

(2) The measure of the central angle which subtends an arc of length 5π cm. in a circle of radius length 15 cm. is $\dots\dots\dots$

- (a) 30° (b) 60° (c) 90° (d) 180°

(3) $f(x) = 12 - 3x$ is negative on the interval $\dots\dots\dots$

- (a) $[-4, \infty[$ (b) $]-\infty, 4[$ (c) $]4, \infty[$ (d) $]-\infty, -4]$

(4) $\sin(90^\circ - \theta) \sec \theta = \dots\dots\dots$

- (a) 1 (b) -1 (c) 0 (d) 90°

2 Complete each of the following :

(1) The quadratic equation whose roots are $4 + 3i$, $4 - 3i$ is $\dots\dots\dots$

(2) In ΔXYZ if $\sin x - \cos z = 0$, then $\sin y = \dots\dots\dots$

(3) If one root of the two roots of the equation $x^2 + 5x + k = 3$ is the multiplicative inverse of the other root , then $k = \dots\dots\dots$

(4) If $\csc(\theta + 20^\circ) = \sec(3\theta + 30^\circ)$ where $\theta \in 0 < \theta < 90^\circ$, then $\cos 6\theta = \dots\dots\dots$

3 [a] If $\frac{6-4i}{1-i} = a + bi$ where $a, b \in \mathbb{R}$, then find the value of : a and b

[b] Without using calculator find the value of : $\sin 150^\circ \cos(-300^\circ) + \cos 930^\circ \cot 240^\circ$

- 4** [a] If L, M are the two roots of the quadratic : $3x^2 - 2x + 5 = 0$
 , then form the quadratic equation whose roots are : $L^2 M, M L^2$

- [b] Find the general solution of the equation : $\cos 2\theta = \sin 4\theta$
 , then find the values of θ where $\theta \in]0, \frac{\pi}{2}[$

- 5** [a] If $f(x) = x^2 - 3x + 2$

(1) Investigate the sign of f

(2) Find in \mathbb{R} the solution set of the inequality : $f(x) \leq 0$

- [b] ABC is an inscribed triangle of a circle whose radius length = 6 cm.
 if $m(\angle A) = 30^\circ$, then find the length of : \widehat{BC}

5

Giza Governorate

El-Haram Educational Zone
 Pyramids Language School



Answer the following questions : (Calculators are permitted)

- 1** Choose the correct answer :

(1) If $x = 3$ is one root of the equation $3x^2 - 8x + m = 0$, then $m = \dots\dots\dots$

- (a) 3 (b) -3 (c) 5 (d) -5

(2) The measure of the central angle subtended an arc of length 2π in a circle of diameter length 12 cm. is equal to $\dots\dots\dots$

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{5}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

(3) The quadratic equation whose two roots are 8 and -13 is $\dots\dots\dots$

- (a) $x^2 - 5x + 104 = 0$ (b) $x^2 - 5x - 104 = 0$
 (c) $x^2 + 5x - 104 = 0$ (d) $x^2 + 5x + 104 = 0$

(4) If $\sin x < 0$ and $\tan x > 0$, then x lies in the $\dots\dots\dots$ quadrant.

- (a) first (b) second (c) third (d) fourth

- 2** Complete each of the following :

(1) In the triangle ABC , if $m(\angle B) = 60^\circ$, $m(\angle C) = \frac{\pi}{2}$, then $m(\angle A) = \dots\dots\dots^\circ$

(2) If one root of the equation $x^2 - kx + k + 2 = 0$ is twice the other , then $k = \dots\dots\dots$

(3) If A and B are two acute angles and $\sin A = \cos B$, then $\sin(A + B) = \dots\dots\dots$

(4) The two functions $f : f(x) = x^2 - 2x + 1$ and $n : n(x) = x - 3$ are positive together at $x \in \dots\dots\dots$

- 3 [a]** If L and M are the two roots of the equation : $2x^2 + 3x - 5 = 0$
Find the quadratic equation whose two roots are : 2 L and 2 M

[b] Without using calculator find the value of :

$$\cos 570^\circ \cos 330^\circ - \cos (-240^\circ) \sin (-150^\circ)$$

- 4 [a]** Find the solution set of the inequality : $x^2 - 4 \geq 0$

[b] If $90^\circ < \theta < 180^\circ$ and $\sin \theta = \frac{4}{5}$

, find the value of : $\sin (90^\circ - \theta) \sin (180^\circ + \theta) \cos^2 (360^\circ - \theta)$

- 5 [a]** Investigate the sign of the function f where $f(x) = 3x - x^2$

[b] If an inscribed angle of measure 40° is subtends an arc of length 6 cm.
, find the circumference of its circle to the nearest cm.

6

Alexandria Governorate

East Educational Zone
Mathematics Directed
(A)



Answer the following questions :

- 1** Choose the correct answer from those given :

(1) The simplest form of the imaginary number i^{30} is

- (a) i (b) 1 (c) -1 (d) -i

(2) The radian measure of the angle $64^\circ 48'$ is

- (a) 0.81 (b) 0.36π (c) 0.18π (d) 0.36

(3) The quadratic equation whose two roots are real and equal and one of two roots is multiplicative inverse of the other is

- (a) $x^2 - 1 = 0$ (b) $x^2 - 6x + 9 = 0$
(c) $2x^2 - 5x + 2 = 0$ (d) $x^2 + 2x + 1 = 0$

(4) If $\sin 2\theta = \cos 4\theta$ where θ is the positive acute angle , then $\tan (90^\circ - 3\theta) = \dots\dots\dots$

- (a) -1 (b) $\frac{1}{\sqrt{3}}$ (c) 1 (d) $\sqrt{3}$

- 2** Complete the following :

(1) If one of the two roots of the equation $x^2 - 3x + c = 0$ is twice the other
, then c =

(2) The range of the function $f(\theta) = 2 \sin \theta$ is

(3) The function f where $f(x) = 3 - x$ is negative in interval

(4) If $\tan \theta = 1.8$ where $90^\circ \leq \theta \leq 360^\circ$, then $m(\angle \theta) = \dots\dots\dots^\circ$

- 3 [a]** Without using calculator prove that : $\sin 60^\circ \cos 330^\circ - \cos 120^\circ \sin 210^\circ = \sin^2 \frac{\pi}{4}$

[b] If $x = \frac{26}{5-i}$, $y = \frac{6+4i}{1+i}$ Prove that : x , y are conjugate and find the value of xy

- 4 [a]** If L, M are two roots of the equation : $x^2 - 7x + 3 = 0$

Form the equation whose two roots are : $2L, 2M$

- [b]** If the terminal side of the angle θ in the standard position intersects the unit circle at the point $(-x, y)$ where $x > 0$, then find the value of : $\tan^2 \theta - \sin \theta \cos \theta$

- 5 [a]** Find the solution set of the inequality :

$x^2 + x - 12 > 0$ and represent it on number line.

- [b]** If $\sin x = \frac{4}{5}$ where $90^\circ < x < 180^\circ$ Find the value of :

$\sin(180^\circ - x) + \tan(360^\circ - x) + 2 \sin(270^\circ - x)$ (Without using calculator)

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Alexandria Governorate

El-Agamy Educational Zone
Maths Inspection



Answer the following questions : (Calculators are permitted)

- 1** Choose the correct answer :

(1) If $x = -1$ is one of the two roots of the equation $x^2 - ax - 2 = 0$, then $a = \dots\dots\dots$

- (a) 2 (b) -2 (c) 1 (d) -1

(2) If $\sin \theta = -1$ and $\cos \theta = \text{zero}$, then θ equals $\dots\dots\dots$

- (a) 90° (b) 180° (c) 270° (d) 360°

(3) If $0^\circ < \theta < 90^\circ$ and $\sin(5\theta) = \cos(4\theta)$, then $m(\angle \theta) = \dots\dots\dots^\circ$

- (a) 14 (b) 18 (c) 12 (d) 10

(4) The expression $(13 - 2i) - (3 - i)$ in the form of the number $a + bi$ is $\dots\dots\dots$

- (a) $10i$ (b) $-10i$ (c) $10 - i$ (d) $10 + i$

- 2** Complete :

(1) The simplest form of the expression $\sin(180^\circ + \theta) + \cos(90^\circ + \theta) = \dots\dots\dots$

(2) The function f where $f(x) = 3 - 2x$ is positive when $x \in \dots\dots\dots$

(3) If $\cos \theta = \frac{\sqrt{3}}{2}$, where $\theta \in]0, 2\pi[$, then the greatest positive value of $\theta = \dots\dots\dots$

(4) If the sum of the two roots of the equation $x^2 - ax + 6 = 0$ equals 5, then $a = \dots\dots\dots$

- 3 [a]** If $\frac{2}{L}$ and $\frac{2}{M}$ are the two roots of the equation : $4x^2 + 3x - 2 = 0$

Form the quadratic equation whose two roots are : L and M

- [b]** A central angle of measure θ in a circle of a radius length 18 cm. and subtends an arc of length 26 cm., find θ in degree measure.

4 [a] Find in \mathbb{R} the solution set of the inequality : $x^2 - 5x - 6 > 0$

[b] If $\sin \theta = \frac{4}{5}$ where $\theta \in \left] \frac{\pi}{2}, \pi \right[$, find the value of the : $2 \sin 150^\circ \cos (-120^\circ) + 4 \tan \theta$

5 [a] Put the number $\frac{2-3i}{3+2i}$ in the form of $a + bi$ where $i^2 = -1$

[b] Investigate the sign of the function f where $f(x) = 8x - x^2 - 15$ in the interval $[2, 6]$

[c] If $\sin \theta = \sin 750^\circ \cos 300^\circ + \sin (-60^\circ) \cot 120^\circ$, where $0^\circ < \theta < 2\pi$

Find : $m(\angle \theta)$

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El-Kalyoubia Governorate

Maths Inspection



Answer the following questions :

1 Complete the following :

(1) The range of the function f where $f(\theta) = \sin \theta$ is

(2) The simplest form of the imaginary number $i^{43} = \dots\dots\dots$

(3) The smallest positive measure of the angle whose measure -690° is

(4) The sign of the function f where $f(x) = 2x - 6$ is negative in the interval

2 Choose the correct answer from those given :

(1) The two roots of the equation $x^2 - 4x + k = 0$ are equal if $k = \dots\dots\dots$

(a) 1

(b) 4

(c) 8

(d) 6

(2) If $\csc(\theta) = 2$ where θ is the measure of an acute angle, then measure of angle θ equals

(a) 15°

(b) 30°

(c) 45°

(d) 60°

(3) The solution set of the equation $x^2 = x$ in \mathbb{R} is

(a) $\{0\}$

(b) $\{1\}$

(c) $\{-1, 1\}$

(d) $\{0, 1\}$

(4) $\sin 2\theta = \cos 4\theta$ where θ is a positive acute angle, then $\tan(90^\circ - 3\theta)$ equals

(a) -1

(b) $-\sqrt{3}$

(c) 1

(d) $\sqrt{3}$

3 [a] Find the value of x and y which satisfy the equation : $\frac{(2+i)(2-i)}{3+4i} = x + iy$

[b] A central angle of measure 150° and subtends an arc length 11 cm. calculate its radius length to the nearest tenth.

4 [a] Prove without using the calculator that : $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ = \sin^2 45^\circ$

[b] Find the solution set of the inequality : $x^2 + 3x \leq 4$ in \mathbb{R}

- 5 [a]** If L and M are the two roots of the equation : $x^2 - 7x + 3 = 0$
 , then find the quadratic equation whose roots are : $L + 2$, $M + 2$

- [b]** If $\tan \theta = \frac{3}{4}$ where $180^\circ < \theta < 270^\circ$
 , then find the value of : $\cos (360^\circ - \theta) - \cos (270^\circ - \theta)$

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El-Monofia Governorate

El-Monofia Educational Directorate
 Mathematics Supervision



Answer the following questions :

- 1** Choose the correct answer :

(1) $\cot (\theta - \pi) - \tan (90^\circ - \theta) = \dots\dots\dots$

- (a) zero (b) 2 (c) 3 (d) 1

(2) If L and M are two roots of the equation $9 - 2x^2 - 8x = 0$, then $L^2 + M^2 = \dots\dots\dots$

- (a) -4 (b) 25 (c) 7 (d) 64

(3) The arc length in a circle of radius 6 cm. and that opposite to a central angle of
 measure $\frac{\pi}{3} = \dots\dots\dots$ cm.

- (a) $\frac{3\pi}{2}$ (b) 6π (c) $\frac{5\pi}{2}$ (d) 2π

(4) If $f(x) = 3 \sin x$, for each $x \in \mathbb{R}$, then the maximum possible value of the function
 $f(x)$ is $\dots\dots\dots$

- (a) -3 (b) 1 (c) 3 (d) zero

- 2** Complete each of the following :

(1) The solution set of the equation $(x + 2)^2 = 25$ in \mathbb{R} is $\dots\dots\dots$

(2) The quadratic equation whose two roots are $1 + i$ and $1 - i$ is $\dots\dots\dots$

(3) If $\cos \theta = 0.5$, $\theta \in [\pi, 2\pi]$, then $m(\angle \theta) = \dots\dots\dots^\circ$

(4) The function f where $f(x) = x + 3$ is positive , for each $x \in \dots\dots\dots$

- 3 [a]** Find the value of : x and y if $x + yi = \frac{3 + 4i}{5 - 2i}$

[b] If the angle θ is in standard position and its terminal side intersects the unit circle at
 the point $(x, \frac{3}{5})$ where $x > 0$ Find the basic trigonometric function.

- 4 [a]** Without using calculator find the value of : $3 \sin 150^\circ \tan 585^\circ + \sin 270^\circ \cos^2 135^\circ$

[b] If L and M are roots of the equation : $2x^2 - 3x - 7 = 0$
 , form the quadratic equation of two roots : $2L - 3$ and $2M - 3$

- 5 [a] If $13 \sin B - 12 = 0$ where $B \in]90^\circ, 180^\circ[$, then find the value of : $\cos (180^\circ - B) \csc (270^\circ + B) \cot (90^\circ - B)$
- [b] Investigate the sign of function $f : f(x) = 6 - 5x - x^2$, then find in \mathbb{R} the S.S. of : $x^2 + 5x - 6 \leq 0$

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El-Dakahlia Governorate

Math Supervision



Answer the following questions :

1 Complete :

- (1) The angle of measure 480° lies in quadrant.
- (2) If sign of the function $f(x) = x^2 + bx + c$ is positive in \mathbb{R} , then b^2
- (3) If $a = 3 + \sqrt{2}i$, $ab = 11$, then $b =$
- (4) The radian measure of inscribed angle opposite to arc of length = its diameter =

2 Choose the correct answer :

- (1) The smallest positive angle satisfies $2 \sin A = \sqrt{3}$ is
 (a) 120° (b) 60° (c) 45° (d) 150°
- (2) If $(2 - i)$ is a root of equation $x^2 + bx + 5 = 0$, then $b =$
 (a) $2 + i$ (b) 5 (c) -4 (d) $-2i$
- (3) The sign of function $f(x) = x^2 + 2$ is positive in
 (a) \mathbb{R} (b) \mathbb{R}_+ (c) $\mathbb{R} - \{0\}$ (d) $\mathbb{R} - \{2\}$
- (4) If $5 \sin B = 3$, $\frac{\pi}{2} < B < \pi$, then $\tan B =$
 (a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $-\frac{4}{3}$ (d) $\frac{4}{5}$

- 3 [a] If $L + 2$, $M + 2$ are two roots of the equation : $x^2 - 5x + 3 = 0$, find the equation whose roots are : L^2, M^2

[b] Find the general solution of the equation : $\sin(3x) \times \sec(6x) = \tan 225^\circ$

- 4 [a] Find S.S. of the inequality : $x^2 - 5x \leq 6$

[b] If $3 \cot \theta + 4 = 0$, $\theta \in]\frac{\pi}{2}, \pi[$

Find the value of : $5 \sin\left(\frac{\pi}{2} + \theta\right) \cos 300^\circ + 3 \operatorname{cosec}(\pi + \theta) \tan 135^\circ$

- 5 [a] Find in the simplest form : $(1 + 2i^3)(2 + 3i^5 + 4i^6)$

[b] Find the value of a which makes one of the roots of the equation :

$$x^2 - ax + 2a - 4 = 0 \text{ is four times of the other root.}$$



Answer the following questions :

1 Choose the correct answer :

(1) If $\sin 2\theta = \cos 4\theta$ where θ is positive acute angle , then $\tan (90^\circ - 3\theta) = \dots\dots\dots$

- (a) -1 (b) $\frac{1}{\sqrt{3}}$ (c) 1 (d) $\sqrt{3}$

(2) The function $f(x) = 6 - 2x$ is positive at $x \in \dots\dots\dots$

- (a) $]-\infty, 3[$ (b) $]-\infty, 3]$ (c) $]3, \infty[$ (d) $[3, \infty[$

(3) If $\cos \theta = \frac{1}{2}$, $\sin \theta = \frac{-\sqrt{3}}{2}$, then the measure of the angle θ equals $\dots\dots\dots$

- (a) $\frac{2\pi}{3}$ (b) $\frac{5\pi}{6}$ (c) $\frac{5\pi}{3}$ (d) $\frac{11\pi}{6}$

(4) $(1+i)^{10} = \dots\dots\dots$ in the simplest form.

- (a) $10i$ (b) $32i$ (c) $-32i$ (d) $16i$

2 Complete :

(1) If $x=3$ is one of the roots of the equation $x^2 - mx - 27 = 0$, then $m = \dots\dots\dots$

(2) The measure of the inscribed angle whose radius 12 cm. and its arc length is 18 cm. is $\dots\dots\dots$ in degree.

(3) The angle whose measure 930° lies in the $\dots\dots\dots$ quadrant.

(4) The S.S. of $x^2 + 9 = 0$ in the complex numbers is $\dots\dots\dots$

3 [a] If one of the roots of the quadratic equation : $4kx^2 + 7x + k^2 + 4 = 0$ is the multiplicative inverse to the other , then find the value of : k

[b] If $4 \sin A - 2 \tan A \tan \left(\frac{3\pi}{2} - A \right) = 0$ where $A \in]0, \frac{3\pi}{2}[$, find the measure of : A

4 [a] If L and M are the two roots of the quadratic equation : $x^2 + 3x - 5 = 0$, write the equation whose roots are : L^2 and M^2

[b] Find the S.S. in \mathbb{R} of the inequality : $x(x+2) - 3 \leq 0$

5 [a] If $x + 2yi - 3y = (3 - 2i)^2$ Find the value of : x and y

[b] If θ is a central angle in its standard position and $B\left(x, \frac{3}{5}\right)$ is the intersection point of its terminal side with the unit circle , find the value of :
 $\sin(90^\circ + \theta) - \cot(180^\circ + \theta) \cos(90^\circ + \theta)$

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Maths Inspection
Language Schools



Answer the following questions : (Calculators are allowed)

1 Choose the correct answer :

(1) If L and M are the two roots of the equation $9 - 2x^2 - 8x = 0$, then $L^2 + M^2 = \dots\dots\dots$

- (a) -4 (b) 25 (c) 7 (d) 64

(2) If $2 \cos \theta = \sqrt{3}$, and $\pi < \theta < \frac{3\pi}{2}$, then $m(\angle \theta) = \dots\dots\dots$

- (a) $\frac{\pi}{3}$ (b) $\frac{6\pi}{7}$ (c) $\frac{4\pi}{3}$ (d) $\frac{7\pi}{6}$

(3) If L and $2 - L$ are the two roots of the equation $x^2 + (k - 3)x + 6 = 0$, then $k = \dots\dots\dots$

- (a) 1 (b) 2 (c) 3 (d) 5

(4) The range of function f where $f(\theta) = \frac{3}{2} \sin \theta$ is $\dots\dots\dots$

- (a) $\left\{-\frac{3}{2}, \frac{3}{2}\right\}$ (b) $\left] \text{zero}, \frac{3}{2} \right]$ (c) $\left[-\frac{3}{2}, \frac{3}{2}\right]$ (d) $\left]-\frac{3}{2}, \frac{3}{2} \right[$

2 Complete :

(1) The simplest form of the imaginary number i^{43} is $\dots\dots\dots$

(2) The angle whose measure is (930°) is located at the $\dots\dots\dots$ quadrant.

(3) The function $f: [-4, 7] \longrightarrow \mathbb{R}$ where $f(x) = 6 - 2x$ has a positive sign in the interval $\dots\dots\dots$

(4) If $\cos(90^\circ + \theta^\circ) + \sin(90^\circ + 2\theta^\circ) = 0$, where $0 < \theta^\circ < 45^\circ$, then $\sin 2\theta^\circ = \dots\dots\dots$

3 [a] If $\frac{1}{L}$ and $\frac{1}{M}$ are the two roots of the equation : $6x^2 - 5x + 1 = 0$

, then form the quadratic equation whose two roots are : L and M

[b] If $\sin \theta = \sin 750^\circ \cos 300^\circ + \sin(-60^\circ) \cot 120^\circ$ where $0^\circ < \theta < 2\pi$ Find : $m(\angle \theta)$

4 [a] If $f: \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = 8x - x^2 - 15$

(1) Graph the function curve in the interval $[1, 7]$

(2) Determine the sign of the function.

[b] Find the value of x and y which satisfy the equation : $\frac{(2+i)(2-i)}{3+4i} = x + yi$

5 [a] Solve the inequality : $(x+3)^2 \leq 10 - 3(x+3)$ in \mathbb{R}

[b] (1) If $4 \sin A - 3 = 0$, find : $m(\angle A)$ where $A \in]0, \frac{\pi}{2}[$

(2) If $\sin \alpha = \frac{4}{5}$ where $90^\circ < \alpha < 180^\circ$

, find : $\sin(180^\circ - \alpha) + \tan(360^\circ - \alpha) + 2 \sin(270^\circ - \alpha)$

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El-Fayoum Governorate

Directorate of Education
Supervision of Mathematics



Answer the following questions : (Calculators are permitted)

1 Choose the correct answer :

(1) The solution set of the equation $x^2 - x = 0$ in \mathbb{R} is

- (a) $\{1, -1\}$ (b) $\{0\}$ (c) $\{1, 0\}$ (d) \emptyset

(2) The angle of measure 60° in the standard position is equivalent to the angle of measure

- (a) 120 (b) 240 (c) 300 (d) 420

(3) If one root of the equation $x^2 - 3x + 2 = 0$ is the multiplicative inverse of the other root, then $a = \dots\dots\dots$

- (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 2 (d) 3

(4) If $\sin \theta = -1$, $\cos \theta = 0$, then the measure of angle $\theta = \dots\dots\dots^\circ$

- (a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

2 Complete :

(1) The quadratic equation whose roots are 2 and 3 is

(2) If the two roots of the quadratic equation $3x^2 - 6x + k = 0$ are equal, then $k = \dots\dots\dots$

(3) $\sin 25^\circ = \cos \dots\dots\dots^\circ$

(4) The range of the function f where $f(x) = 2 \sin \theta$ is

3 [a] If L and M are the two roots of the equation : $x^2 - 7x + 3 = 0$

, then form the quadratic equation whose roots are : $2L$ and $2M$

[b] The measure of central angle is 105° and subtend arc of length $\frac{7\pi}{3}$ cm.

Find length of the diameter of the circle.

4 [a] Draw the curve of the function $f : f(x) = x^2 - 9$ in the interval $[-3, 4]$, from the graph determine the sign of f in that interval.

[b] Find the value of θ where $\theta \in]0, \frac{\pi}{2}[$, which satisfies the equation :

$$2 \cos\left(\frac{\pi}{2} - \theta\right) = 1$$

5 [a] If $x = \frac{13}{5-i}$, $y = \frac{3+2i}{1+i}$, prove that : x, y are two conjugates numbers.

[b] Without using calculator prove that : $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ = \sin^2 \frac{\pi}{4}$

14**El-Menia Governorate**

Maths Department
El-Menia Official Language School



Answer the following questions : (Calculator is allowed)

1 Choose the correct answer of those given :

(1) The measure of the central angle in a circle of radius length 15 cm. and opposite to an arc of length $5\pi = \dots\dots\dots$

- (a) 30° (b) 60° (c) 90° (d) 180°

(2) If one of the two roots of the equation $x^2 - (b-3)x + 5 = 0$ is the additive inverse of the other root, then $b = \dots\dots\dots$

- (a) 5 (b) 3 (c) -5 (d) -3

(3) The simplest form of i^{43} is $\dots\dots\dots$

- (a) 1 (b) -1 (c) i (d) $-i$

(4) The function $f : f(x) = 5x - 3$ is positive at $\dots\dots\dots$

- (a) $x > \frac{3}{5}$ (b) $x < \frac{3}{5}$ (c) $x > \frac{5}{3}$ (d) $x < \frac{-5}{3}$

2 Complete :

(1) $(4-3i)(4+3i) = \dots\dots\dots$

(2) The solution set of the equation $x^2 + 9 = 0$ in \mathbb{R} is $\dots\dots\dots$

(3) If $\sin 2\theta = \cos 4\theta$ where θ is a positive acute angle, then $\tan(90^\circ - 3\theta) = \dots\dots\dots$

(4) The angle of measure 750° lies on the $\dots\dots\dots$ quadrant.

3 [a] Solve in the set of complex number : $2x^2 + 6x + 5 = 0$

[b] Without using the calculator, prove that :

$$\sin 600^\circ \cos(-30^\circ) + \sin 150^\circ \cos(-240^\circ) = -1$$

4 [a] If L, M are the roots of equation : $x^2 - 7x + 6 = 0$

Form the equation whose roots are : L^2, M^2

[b] Find x and all trigonometric function of θ drawn in unit circle its coordinates $(-x, x)$, $x > 0$

5 [a] If 2, 5 are the roots of the equation : $x^2 + ax + b = 0$ Find : a, b

[b] Form the quadratic equation whose two roots are : $\frac{-2+2i}{1+i}, \frac{-2-4i}{2-i}$

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Qena Governorate

Qena Educational Zone
Math Supervision

Answer the following questions : (Calculators are permitted)

1 Complete :

- (1) The quadratic equation in the set of the complex numbers whose roots are $-2i, 2i$ is
- (2) The function $f : [-4, 7] \rightarrow \mathbb{R}$ where $f(x) = 6 - 2x$ has a positive sign in the interval
- (3) If $4 \sin^2 \theta - 3 = 0$, where $\theta \in [0^\circ, 90^\circ]$, then $m(\angle \theta) = \dots^\circ$
- (4) The range of the function $f(x) = 2 \sin 3x$ is

2 Choose the correct answer from the given ones :

- (1) The simplest form of the imaginary number $i^{23} = \dots$
 (a) -1 (b) 1 (c) $-i$ (d) i
- (2) If $\angle A$ and $\angle B$ are two acute angles where $\sin A = \cos B$, then $\sin(A + B) = \dots$
 (a) -1 (b) 1 (c) 0 (d) 90°
- (3) The length of the arc that is subtended by a central angle of measure 210° in a circle of diameter length 12π cm. is cm.
 (a) 14π (b) 14 (c) $7\pi^2$ (d) 7
- (4) If one root of the equation $2x^2 + (a - 2)x - 7 = 0$ is equal to the additive inverse of the other , then the value of $a = \dots$
 (a) 2 (b) -2 (c) 7 (d) 0

3 [a] (1) Determine the sign of the function $f : f(x) = -x^2 + 7x - 10$

(2) Find the solution set of the inequality in $\mathbb{R} : x^2 + 3x - 4 \leq 0$

[b] If $180^\circ < \theta < 270^\circ$, where $\cos \theta = -\frac{4}{5}$

Find the value of : $\sin \theta \cos (180^\circ - \theta) + \cos (-\theta) \sin (\theta - 270^\circ)$

4 [a] If L, M are the roots of the equation : $x(2x - 3) = 5$

Find the equation whose roots are : L^2, M^2

[b] Without using calculator find the value of : $\sin 120^\circ \cos 330^\circ - \cos 420^\circ \sin (-30^\circ)$

- 5 [a] If $x = 3 + 2i$ and $y = \frac{4-2i}{1-i}$, then find $x + y$ in the form of a complex number.

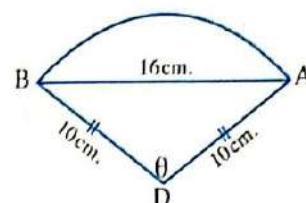
[b] In the opposite figure :

\widehat{AB} is an arc in a circle of radius 10 cm.

and $AB = 16$ cm.

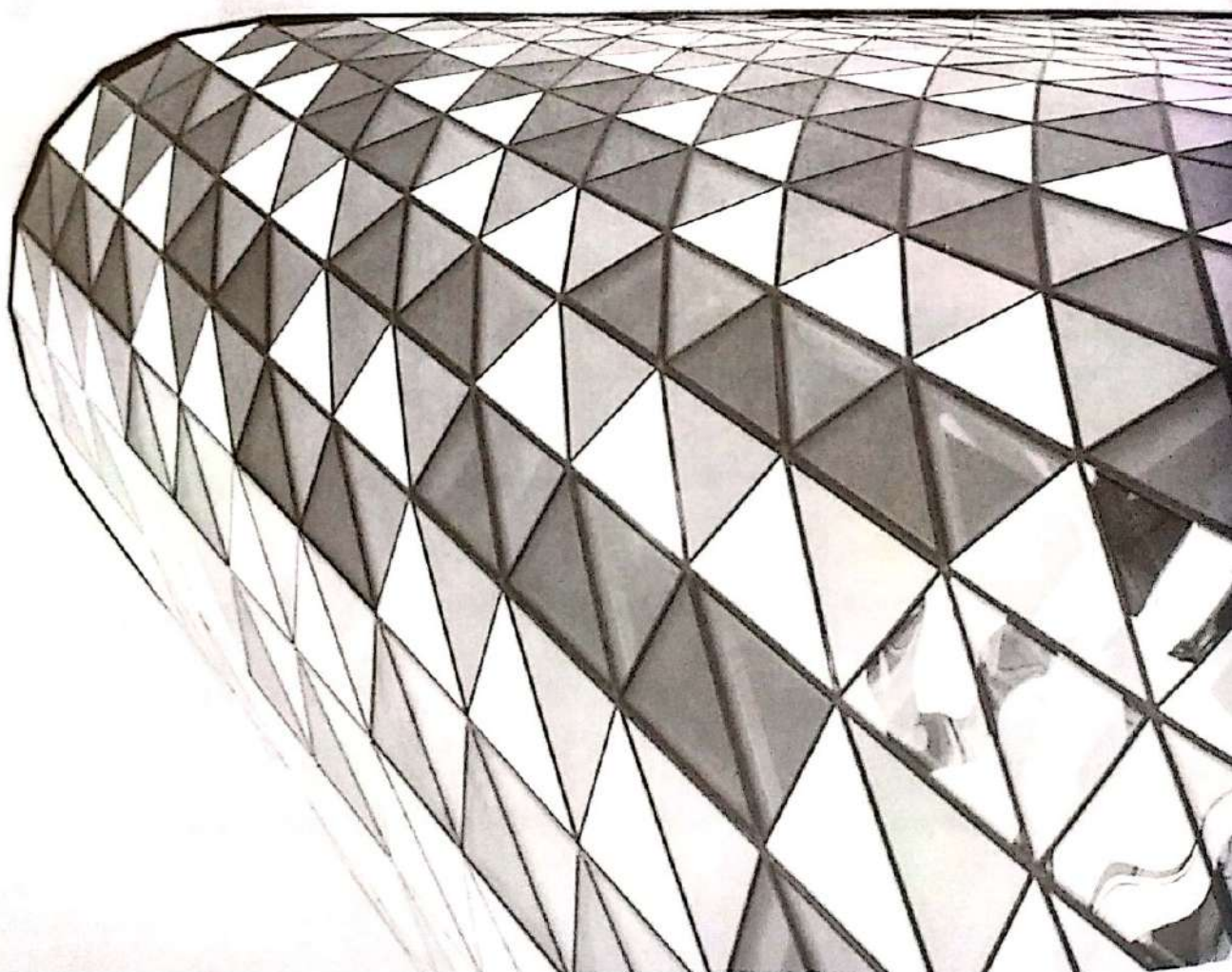
Find θ in radian measure

, then find the length of the arc : \widehat{AB}

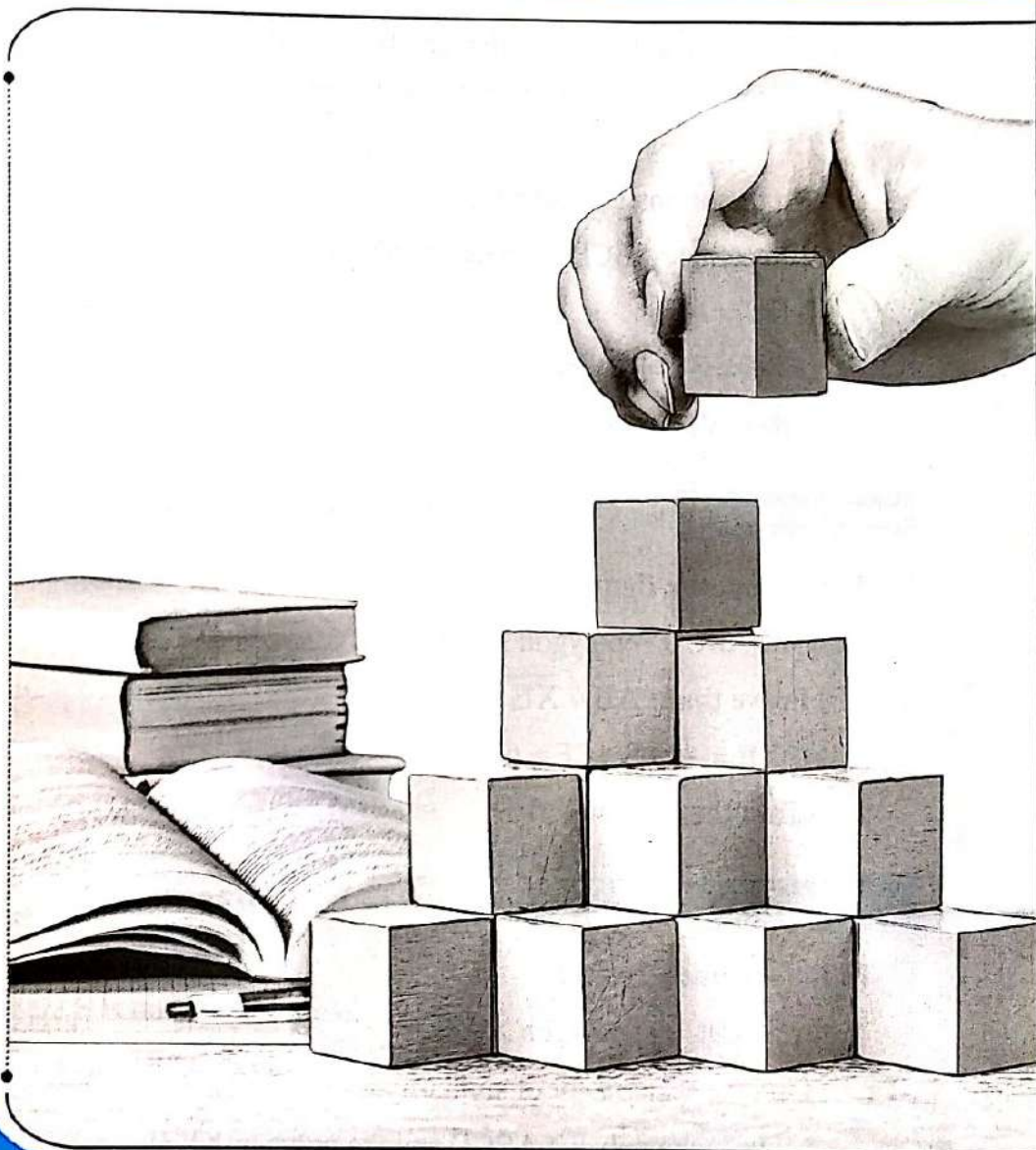


Second

Geometry



Accumulative Tests on Geometry





Test

1

on lesson 1 – unit 3

Total mark

10

Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Complete the following :

- (1) Two polygons of the same number of sides are similar if
- (2) If the scale factor of similarity of two polygons = 1 , then the two polygons are
- (3) Two similar polygons , the ratio between the lengths of two corresponding sides in them is 2 : 3 , if the perimeter of the smaller is 14 cm. , then the perimeter of the bigger is cm.

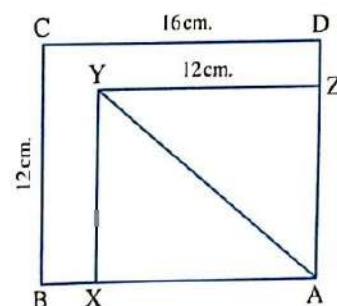
(4) In the opposite figure :

If rectangle ABCD ~ rectangle AXYZ

, DC = 16 cm.

, BC = ZY = 12 cm.

, then AY = cm.



Second question

4 marks

(1) 2 marks

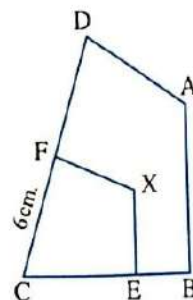
(2) 2 marks

In the opposite figure :

Polygon ABCD ~ polygon XECF

(1) Prove that : $\overline{AB} \parallel \overline{XE}$

(2) If $XE = \frac{1}{2} AB$, CF = 6 cm.
 , find the length of : \overline{FD}



Third question

4 marks

(1) 2 marks

(2) 2 marks

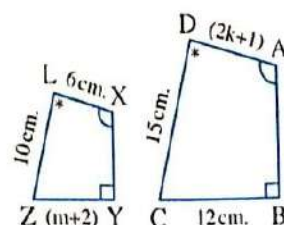
In the opposite figure :

Polygon ABCD ~ polygon XYZL

(1) Find the scale factor of similarity

between the polygon ABCD and the polygon XYZL

(2) Find the value of each of : m , k





Answer the following questions :

First question

4 marks

each item 1 mark

Choose the correct answer from those given :

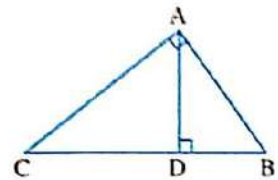
- (1) Two similar rectangles , the two dimensions of the first are 12 cm. , 8 cm. and the perimeter of the second is 60 cm. , then the length of the second rectangle is

(a) 12 cm. (b) 18 cm. (c) 24 cm. (d) 16 cm.

- (2) In the opposite figure :

Which of the following expressions is wrong ?

- (a) $(AB)^2 = BD \times DC$ (b) $(AC)^2 = CD \times CB$
(c) $(AD)^2 = DB \times DC$ (d) $AB \times AC = BC \times AD$

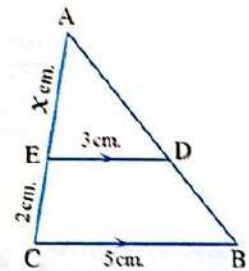


- (3) In the opposite figure :

If $\overline{DE} \parallel \overline{BC}$

, then $x = \dots\dots\dots$

- (a) 6 cm. (b) 3 cm.
(c) 5 cm. (d) 1.2 cm.



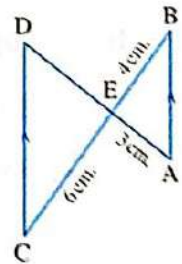
- (4) In the opposite figure :

$\overline{AB} \parallel \overline{CD}$, $AE = 3$ cm.

, $BE = 4$ cm. , $EC = 6$ cm.

, then $ED = \dots\dots\dots$

- (a) 4 cm. (b) 6 cm. (c) 3 cm. (d) $4\frac{1}{2}$ cm.



Second question

6 marks

(1) 3 marks

(2) 3 marks

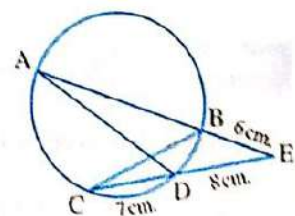
In the opposite figure :

$\overline{AB} \cap \overline{CD} = \{E\}$ where E is outside the circle.

If $EB = 6$ cm. , $ED = 8$ cm. , $DC = 7$ cm.

- (1) Prove that : $\triangle ADE \sim \triangle CBE$

- (2) Find the length of : \overline{AE}





Answer the following questions :

First question

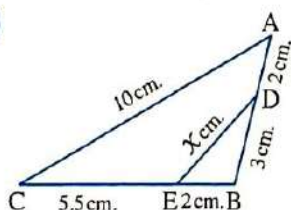
2 marks

each item $\frac{1}{2}$ mark

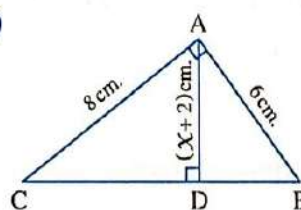
In each of the following figures , find the value of the symbol used in measure.

Explain your answer :

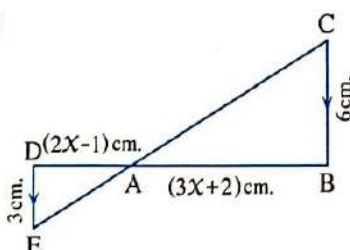
(1)



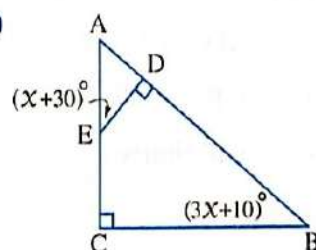
(2)



(3)



(4)



Second question

4 marks

(1) 2 marks

(2) 2 marks

In the opposite figure :

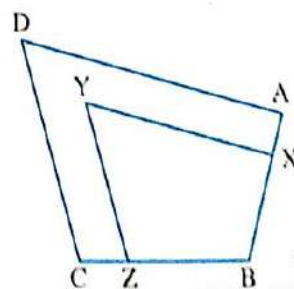
(1) If polygon ABCD ~ polygon XBZY

, prove that : $\overline{XY} \parallel \overline{AD}$

(2) If the perimeter of the polygon ABCD = 18 cm,

, the perimeter of the polygon XBZY = 12 cm,

, $\overline{XB} = 3$ cm, , then find the length of : \overline{AB}



Third question

4 marks

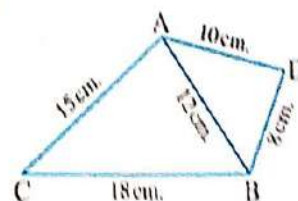
(1) 2 marks

(2) 2 marks

Using the givens in the opposite figure , prove that :

(1) $\triangle ABC \sim \triangle DBA$

(2) \overline{BA} bisects $\angle DBC$





Test

4

till lesson 4 – unit 3

Total mark

10

Answer the following questions :

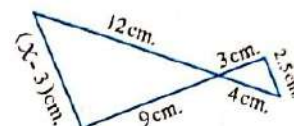
First question

2 marks

each item $\frac{1}{2}$ mark

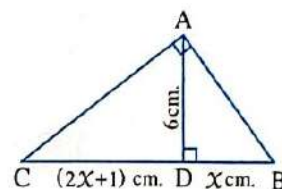
Complete the following :

- (1) If two angles in one triangle are congruent to their corresponding angles in another triangle , then the two triangles are
- (2) If the ratio between the perimeters of two similar polygons is 4 : 9 , then the ratio between their areas is
- (3) In the opposite figure :
- $x = \dots\dots\dots$



- (4) In the opposite figure :

$x = \dots\dots\dots$



Second question

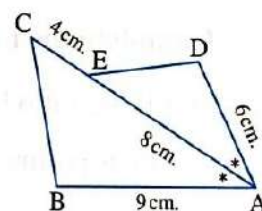
4 marks

In the opposite figure :

\overrightarrow{AE} bisects $\angle DAB$

, the area of $\triangle ADE = 12 \text{ cm}^2$

Find the area of : $\triangle ABC$



Third question

4 marks

ABCD , XYZL are two similar polygons. If M is the midpoint of \overline{BC}

, N is the midpoint of \overline{YZ} , $AM = 4 \text{ cm}$, $XN = 9 \text{ cm}$.

, prove that :

area of polygon ABCD : area of polygon XYZL = 16 : 81

Answer the following questions :

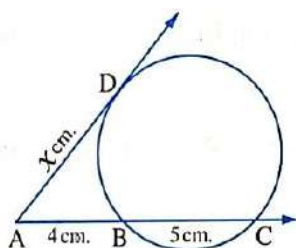
First question

4 marks

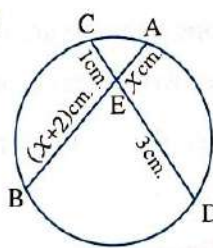
each item 1 mark

Find the numerical value of x in each of the following figures :

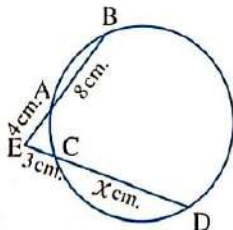
(1)



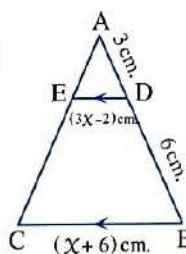
(2)



(3)



(4)



Second question

3 marks

each item 1 mark

Complete the following :

- (1) If the ratio between the areas of two similar triangles is 4 : 9 , then the ratio between their perimeters is
- (2) If each one of two polygons is similar to a third polygon , then the two polygons are
- (3) Two isosceles triangles are similar if

Third question

3 marks

ABC , DEF are two similar triangles , X is the midpoint of \overline{BC} and Y is the midpoint of \overline{EF}

Prove that : $\triangle ABX \sim \triangle DEY$

Answer the following questions :

First question

2 marks

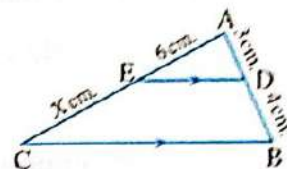
each item $\frac{1}{2}$ mark

Complete the following :

(1) In the opposite figure :

If $\overline{DE} \parallel \overline{BC}$

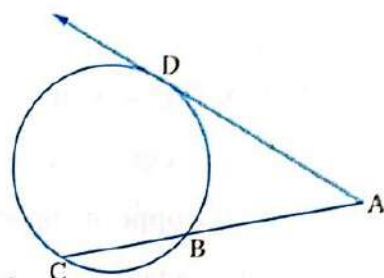
, then $x = \dots\dots\dots$



(2) In the opposite figure :

If \overline{AD} is a tangent to the circle

, then $(AD)^2 = \dots\dots\dots$



(3) The ratio between the areas of two similar triangles equals $\dots\dots\dots$

(4) If a line is drawn parallel to one side of a triangle and intersects the other two sides
 , then it divides them into segments whose lengths are $\dots\dots\dots$

Second question

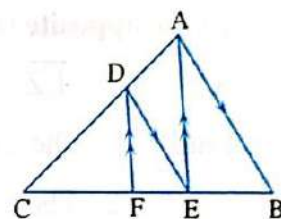
4 marks

In the opposite figure :

ABC is a triangle , $D \in \overline{AC}$

, $\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AE}$

Prove that : $(CE)^2 = CF \times CB$



Third question

4 marks

If the lengths of two corresponding sides in two similar polygons are 6 cm. and 8 cm.

, and the area of the bigger polygon is 144 cm^2

, find the area of the smaller polygon.

**Test****7**

till lesson 2 – unit 4

Total mark

10*Answer the following questions :***First question**

2 marks

each item $\frac{1}{2}$ mark**Complete the following :****(1)** In any right-angled triangle, the altitude to the hypotenuse separates the triangle into**(2)** In the opposite figure :

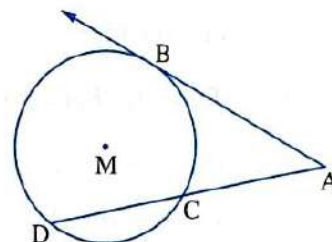
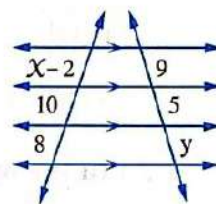
$$x + y = \dots\dots\dots$$

(3) If $\triangle ABC \sim \triangle DEF$, area of $\triangle ABC = 4$ area of $\triangle DEF$ and

DE = 6 cm. , then AB = cm.

(4) In the opposite figure : \overrightarrow{AB} is a tangent to the circle M

if $(AB)^2 = \dots\dots\dots$

**Second question**

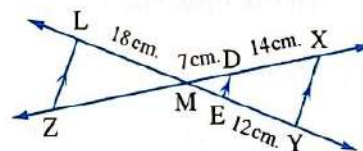
4 marks

(1) 2 marks

(2) 2 marks

In the opposite figure :

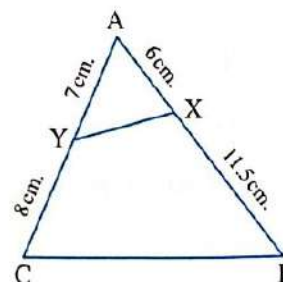
$$\overline{XY} \parallel \overline{DE} \parallel \overline{LZ}$$

Find : **(1)** The length of \overline{EM} **(2)** The length of \overline{MZ} **Third question**

4 marks

(1) 2 marks

(2) 2 marks

In the opposite figure :**Prove that :****(1)** $\triangle ABC \sim \triangle AYX$ **(2)** The figure XBCY is a cyclic quadrilateral.



Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Complete the following :

(1) The interior and exterior bisectors of an angle of a triangle at a vertex are

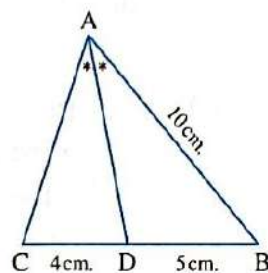
(2) If $\triangle ABC \sim \triangle XYZ$ and $AB = 3 XY$

, then $\frac{\text{The area of } \triangle XYZ}{\text{The area of } \triangle ABC} = \dots\dots\dots$

(3) In the opposite figure :

\overrightarrow{AD} bisects $\angle BAC$

, then $AD = \dots\dots\dots$ cm.

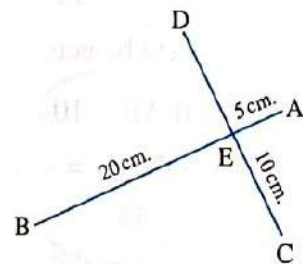


(4) In the opposite figure :

If $\overline{AB} \cap \overline{CD} = \{E\}$, then

the points A, C, B and D lie

on one circle if $ED = \dots\dots\dots$ cm.



Second question

4 marks

(1) 2 marks

(2) 2 marks

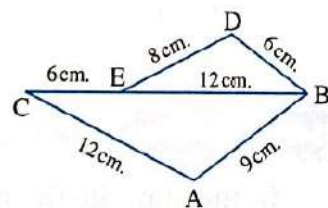
In the opposite figure :

B, E and C are collinear.

Prove that :

(1) $\triangle ABC \sim \triangle DBE$

(2) \overrightarrow{BC} bisects $\angle ABD$



Third question

4 marks

XYZ is a triangle, $\angle XYZ$ is bisected by a bisector which intersects \overline{XZ} at M , then draw $\overrightarrow{MN} \parallel \overline{ZY}$ to intersect \overline{XY} at N

Prove that : $\frac{XY}{YZ} = \frac{XN}{YN}$ and if $XY = 6$ cm. , $YZ = 4$ cm.

, find the length of : \overline{XN}

Answer the following questions :

First question

2 marks

each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) In the opposite figure :

If $\overline{DE} \parallel \overline{BC}$

, then $x = \dots\dots\dots$ cm.

(a) 4

(b) 5

(c) 6

(d) 8

(2) If $\triangle ABC \sim \triangle XYZ$ and $AB = 4 XY$

, then $\frac{\text{The area of } \triangle XYZ}{\text{The area of } \triangle ABC} = \dots\dots\dots$

(a) $\frac{1}{4}$

(b) $\frac{1}{16}$

(c) $\frac{4}{1}$

(d) $\frac{16}{1}$

(3) In the opposite figure :

\overline{AD} bisects $\angle A$, $\frac{BD}{DC} = \frac{5}{3}$

If $AB = 10$ cm. , $AC = (2y - 1)$ cm.

, then $y = \dots\dots\dots$ cm.

(a) 35

(b) 25

(c) 3.5

(d) 2.5

(4) In the opposite figure :

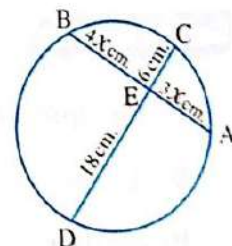
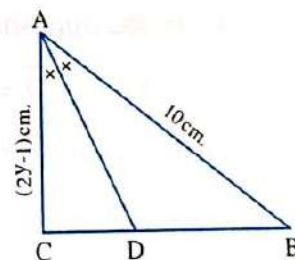
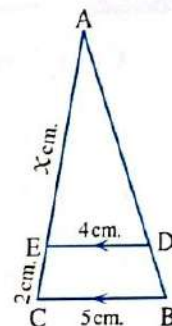
$x = \dots\dots\dots$ cm.

(a) 3

(b) 9

(c) 2

(d) 18



Second question

4 marks

In the opposite figure :

$\overline{BE} \parallel \overline{XY} \parallel \overline{CD}$, $\frac{AB}{AC} = \frac{EY}{YD}$

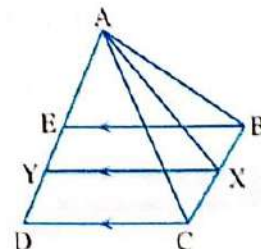
Prove that : \overline{AX} bisects $\angle BAC$

Third question

4 marks

Prove that :

The ratio between the areas of the surfaces of two similar triangles equals the square of the ratio between two corresponding heights of the two triangles.



Answer the following questions :

First question

2 marks

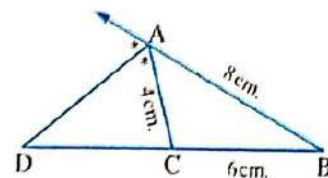
 each item $\frac{1}{2}$ mark

Choose the correct answer from those given :

(1) In the opposite figure :

If \overline{AD} bisects exterior $\angle A$
 , then $CD = \dots\dots\dots$ cm.

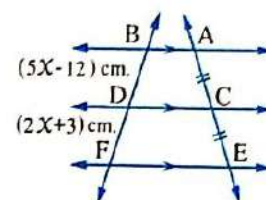
- (a) 2 (b) 6 (c) 4 (d) 8



(2) In the opposite figure :

$X = \dots\dots\dots$ cm.

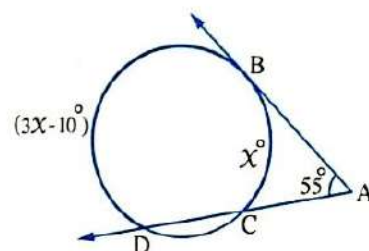
- (a) 5 (b) 3
 (c) 7 (d) 2



(3) In the opposite figure :

If \overline{AB} is a tangent to the circle
 , then $X = \dots\dots\dots$

- (a) 60° (b) 30°
 (c) 15° (d) 55°



(4) If $AM = 4$ cm. , $r = 3$ cm. , such that A is a point outside the circle M

, then $P_M(a) = \dots\dots\dots$

- (a) 16 (b) 9 (c) 25 (d) 7

Second question

4 marks

\overline{AB} , \overline{CD} are two chords in a circle intersecting at E , if E is the midpoint of \overline{AB}
 , $CE = 4$ cm. , $ED = 9$ cm. , find the length of : \overline{AB}

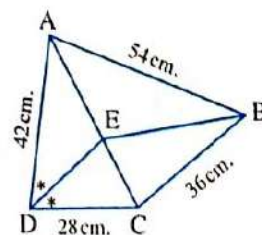
Third question

4 marks

In the opposite figure :

Prove that :

\overline{BE} bisects $\angle ABC$



Final Revision on Geometry

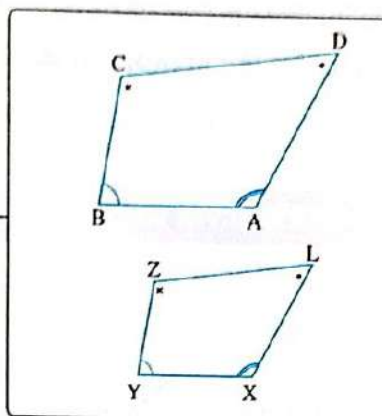


Final Revision on Geometry

Remember The similarity of polygons

Two polygons M_1 and M_2 (having the same number of sides) are said to be similar if the following two conditions satisfied together :

- 1 Their corresponding angles are congruent.



$$\begin{aligned} \text{i.e. } m(\angle A) &= m(\angle X) \\ , m(\angle B) &= m(\angle Y) \\ , m(\angle C) &= m(\angle Z) \\ , m(\angle D) &= m(\angle L) \end{aligned}$$

- 2 The lengths of their corresponding sides are proportional.

$$\text{i.e. } \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX} = K$$

In this case , we say that :

- The polygon $ABCD \sim$ the polygon $XYZL$,
that means the polygon $ABCD$ is similar to the polygon $XYZL$
- K is the scale factor of similarity of the polygon $ABCD$ to the polygon $XYZL$
- $\frac{1}{K}$ is the scale factor of similarity of the polygon $XYZL$ to the polygon $ABCD$

Remarks

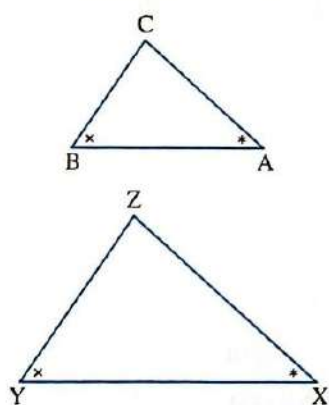
- On writing the similar polygons , write them according to the order of their corresponding vertices.
- If each one of two polygons is similar to a third polygon , then the two polygons are similar.
- All regular polygons which have the same number of sides are similar
(All equilateral triangles are similar , all squares are similar , all regular pentagons are similar , ...)
- If K is the similarity ratio of polygon M_1 to polygon M_2 , and :
If $K > 1$, then polygon M_1 is an enlargement of polygon M_2 , where K is called the enlargement ratio.
If $0 < K < 1$, then polygon M_1 is a shrinking to polygon M_2 , where K is called the shrinking ratio.
If $K = 1$, then polygon M_1 is congruent to polygon M_2
- The ratio between the perimeters of two similar polygons = the ratio between the lengths of two corresponding sides of them.

Remember The similarity of triangles

Two triangles are similar

First case

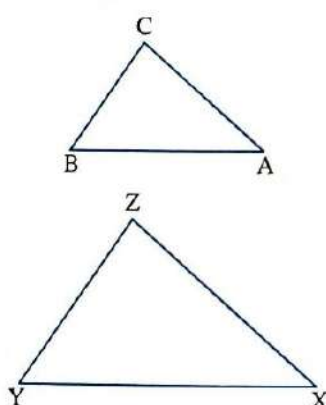
If two angles of one triangle are congruent to their corresponding angles of the other triangle.



If $\angle A \equiv \angle X$
 $\angle B \equiv \angle Y$
 then $\triangle ABC \sim \triangle XYZ$

Second case

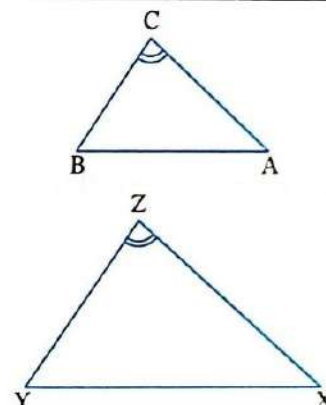
If the side lengths of two triangles are in proportion.



If $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$
 , then $\triangle ABC \sim \triangle XYZ$

Third case

If an angle of one triangle is congruent to an angle of the other triangle and the lengths of the sides including those angles are in proportion.



If $\angle C \equiv \angle Z$
 $\frac{CA}{ZX} = \frac{CB}{ZY}$
 , then $\triangle ABC \sim \triangle XYZ$

Remarks

- Two isosceles triangles are similar if the measure of an angle in one of them is equal to the measure of the corresponding angle in the other triangle.
- Two right-angled triangles are similar if the measure of an acute angle in one of them is equal to the measure of an acute angle in the other triangle.

Corollary

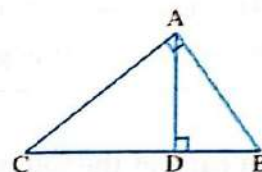
In any right-angled triangle, the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.

In the opposite figure :

If $\triangle ABC$ is a right-angled triangle at A and $\overline{AD} \perp \overline{BC}$

, then $\triangle DBA \sim \triangle DAC \sim \triangle ABC$ and from this we can deduce that :

- $(AB)^2 = BD \times BC$
- $(AC)^2 = CD \times CB$
- $(AD)^2 = BD \times DC$
- $AD \times BC = AB \times AC$



Remember

The relation between the areas of two similar polygons

The ratio between the areas of the surfaces of two similar triangles equals the square of the ratio between the lengths of any two corresponding sides of the two triangles.

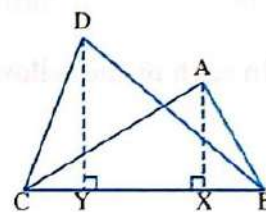
The ratio between the areas of the surfaces of two similar polygons equals the square of the ratio between the lengths of any two corresponding sides of the two polygons.

The ratio of the areas of two triangles having a common base equals the ratio of the two heights of the two triangles.

In the opposite figure :

\overline{BC} is a common base of $\triangle ABC$, $\triangle DBC$

$$\therefore \frac{a(\triangle ABC)}{a(\triangle DBC)} = \frac{\frac{1}{2} BC \times AX}{\frac{1}{2} BC \times DY} = \frac{AX}{DY}$$



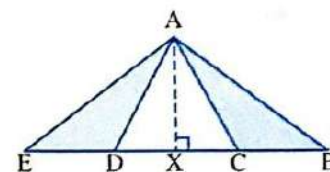
Notice that : It is not necessary that the two triangles are similar.

The ratio of the areas of two triangles having a common height equals the ratio of the lengths of two bases of the two triangles.

In the opposite figure :

\overline{AX} is a common height for $\triangle ABC$, $\triangle ADE$

$$\therefore \frac{a(\triangle ABC)}{a(\triangle ADE)} = \frac{\frac{1}{2} BC \times AX}{\frac{1}{2} DE \times AX} = \frac{BC}{DE}$$



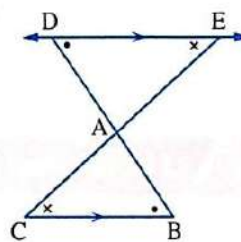
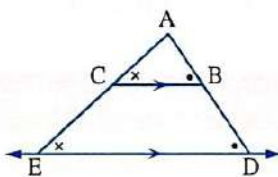
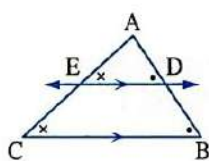
Notice that : It is not necessary that the two triangles are similar.

If a line is drawn parallel to one side of a triangle and intersects the other two sides or the lines containing them , then :

The resulting triangle is similar to the original triangle

It divides them into segments whose lengths are proportional

In each of the following figures :



If $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ and intersects \overleftrightarrow{AB} and \overleftrightarrow{AC} at D and E respectively , then :

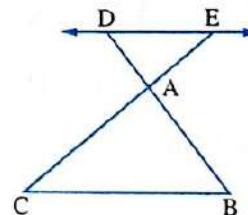
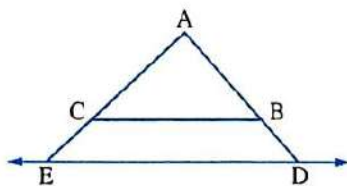
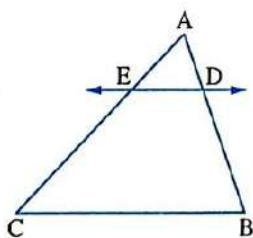
• $\triangle ADE \sim \triangle ABC$

• $\frac{AD}{DB} = \frac{AE}{EC}$ and from the properties of the proportion , we get :

$$\frac{AD}{AB} = \frac{AE}{AC}, \frac{AB}{DB} = \frac{AC}{CE}$$

If a straight line intersects two sides of a triangle and divides them into segments whose lengths are proportional , then it is parallel to the third side of the triangle.

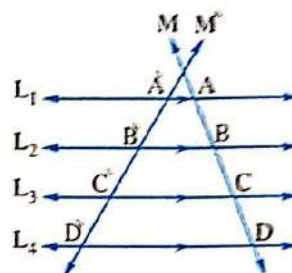
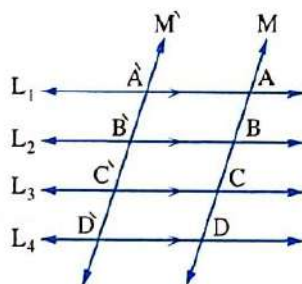
In each of the following figures :



If $\frac{AD}{DB} = \frac{AE}{EC}$, then $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

Remember Talis' theorem

Given several coplanar parallel lines and two transversals , then the lengths of the corresponding segments on the transversals are proportional.



In the previous figures :

If $L_1 \parallel L_2 \parallel L_3 \parallel L_4$ and M, M' are two transversals

, then $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{AC}{A'C'}$

Remember Talis' special theorem

If the lengths of the segments on the transversal are equal , then the lengths of the segments on any other transversal will be also equal.

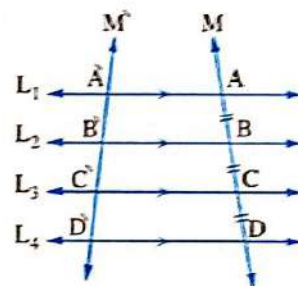
In the opposite figure :

If $L_1 \parallel L_2 \parallel L_3 \parallel L_4$,

M, M' are two transversals to them

and if $AB = BC = CD$

, then $A'B' = B'C' = C'D'$



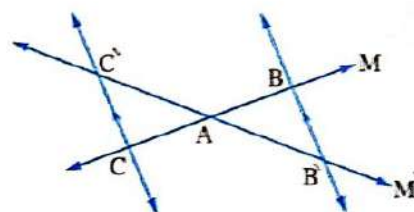
Special case

If the two lines M and M' intersect at

the point A and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CC'}$

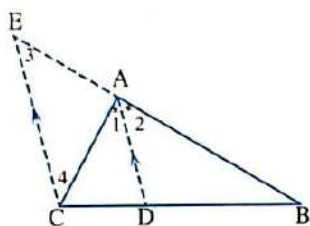
, then $\frac{AB}{AC} = \frac{A'B'}{A'C'}$

and conversely if $\frac{AB}{AC} = \frac{A'B'}{A'C'}$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CC'}$



Theorem

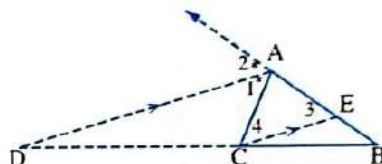
The bisector of the interior or exterior angle of a triangle at any vertex divides the opposite base of the triangle internally or externally into two parts, the ratio of their lengths is equal to the ratio of the lengths of the other two sides of the triangle.



$\therefore \overline{AD}$ bisects $\angle BAC$ internally.

$$\therefore \frac{BD}{DC} = \frac{AB}{AC}$$

$$\therefore AD = \sqrt{AB \times AC - BD \times DC}$$

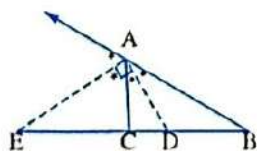


$\therefore \overline{AD}$ bisects $\angle BAC$ externally.

$$\therefore \frac{BD}{DC} = \frac{AB}{AC}$$

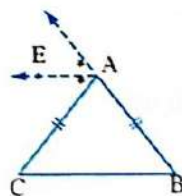
$$\therefore AD = \sqrt{BD \times DC - AB \times AC}$$

The interior and exterior bisectors of the same angle of the triangle are perpendicular.



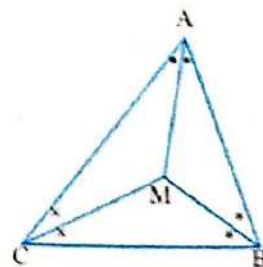
i.e. If \overline{AD} and \overline{AE} are the bisectors of the angle A and the exterior angle of $\triangle ABC$ at A, then $\overline{AD} \perp \overline{AE}$

The exterior bisector of the vertex angle of an isosceles triangle is parallel to the base.

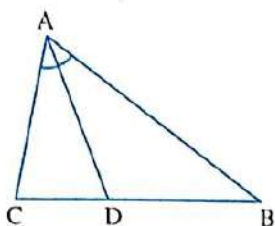


i.e. If $AB = AC$, \overline{AE} bisects the exterior angle at A, then $\overline{AE} \parallel \overline{BC}$

The bisectors of angles of a triangle are concurrent.



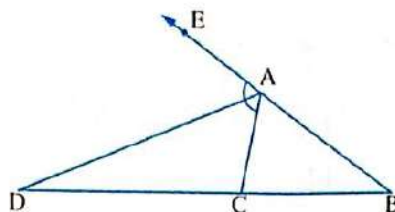
Converse of the theorem



If $D \in \overline{BC}$

such that : $\frac{BD}{DC} = \frac{BA}{AC}$

, then \overrightarrow{AD} bisects $\angle BAC$



If $D \in \overline{BC}$, $D \notin \overline{BC}$

such that : $\frac{BD}{DC} = \frac{BA}{AC}$

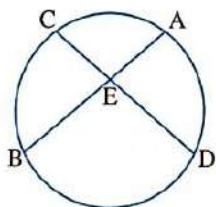
, then \overrightarrow{AD} bisects the exterior angle of $\triangle ABC$ at A

Well known problem and a corollary on it

Well known problem

If \overline{AB} , \overline{CD} are two chords in a circle

, $\overline{AB} \cap \overline{CD} = \{E\}$

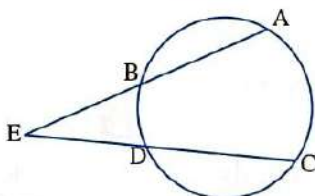


then

$$EA \times EB = EC \times ED$$

If \overline{AB} and \overline{CD} are two chords in a circle

, $\overline{AB} \cap \overline{CD} = \{E\}$

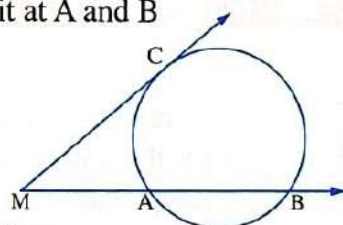


then

$$EA \times EB = EC \times ED$$

Corollary

If M is a point outside the circle, \overrightarrow{MC} touches the circle at C, \overrightarrow{MB} intersects it at A and B



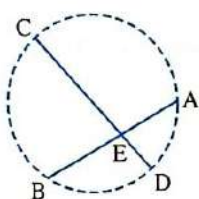
then

$$(MC)^2 = MA \times MB$$

Converse of the well known problem and the corollary

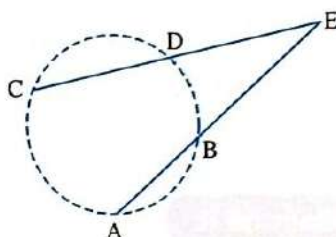
Converse of the well known problem

If $\overline{AB} \cap \overline{CD} = \{E\}$,
A, B, C, D and E are
distinct points and
 $EA \times EB = EC \times ED$



, then the points A, B,
C and D lie on the same
circle.

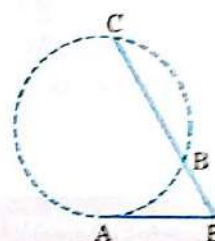
If $\overline{AB} \cap \overline{CD} = \{E\}$,
A, B, C, D and E are
distinct points and
 $EA \times EB = EC \times ED$



, then the points A, B,
C and D lie on the same
circle.

Converse of the corollary

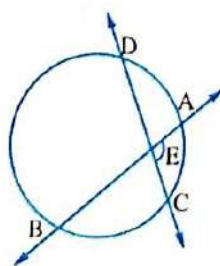
If $E \in \overline{CB}$, $E \notin \overline{BC}$,
and $(EA)^2 = EB \times EC$



, then \overline{EA} is a tangent
segment to the circle
which passes through the
points A, B and C

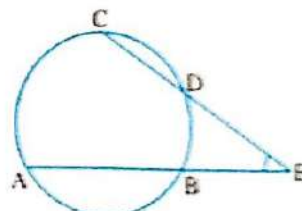
Secant, tangent and measures of angles

- 1** The measure of an angle formed by
two chords that intersect inside
a circle is equal to half the sum of the
measures of the intercepted arcs.



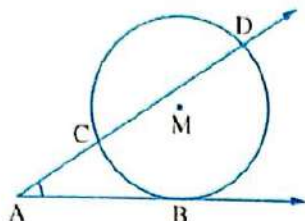
$$m(\angle AEC) = \frac{1}{2} [m(\widehat{AC}) + m(\widehat{BD})]$$

- 2** The measure of an angle formed
by two secants drawn from a point
outside a circle is equal to half the
positive difference of the measures of
the intercepted arcs.



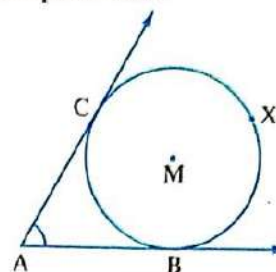
$$m(\angle E) = \frac{1}{2} [m(\widehat{AC}) - m(\widehat{BD})]$$

- 3 The measure of an angle formed by a secant and a tangent drawn from a point outside a circle is equal to half the positive difference of the measures of the intercepted arcs.



$$m(\angle A) = \frac{1}{2} [m(\widehat{BD}) - m(\widehat{BC})]$$

- 4 The measure of an angle formed by two tangents drawn from a point outside a circle is equal to half the positive difference of the measures of the intercepted arcs.



$$m(\angle A) = \frac{1}{2} [m(\widehat{BXC}) - m(\widehat{BC})]$$

Power of a point with respect to a circle

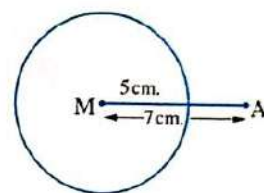
Power of the point A with respect to the circle M in which, the length of its radius r is the real number $P_M(A)$ where $P_M(A) = (AM)^2 - r^2$

For example : In the opposite figure :

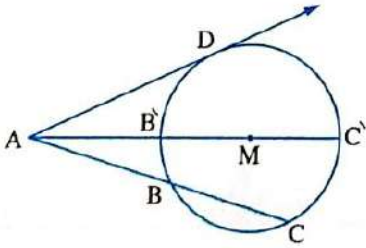
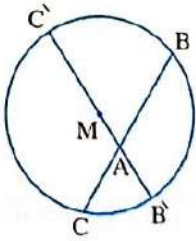
If A is a point outside the circle M

whose radius length equals 5 cm. ,

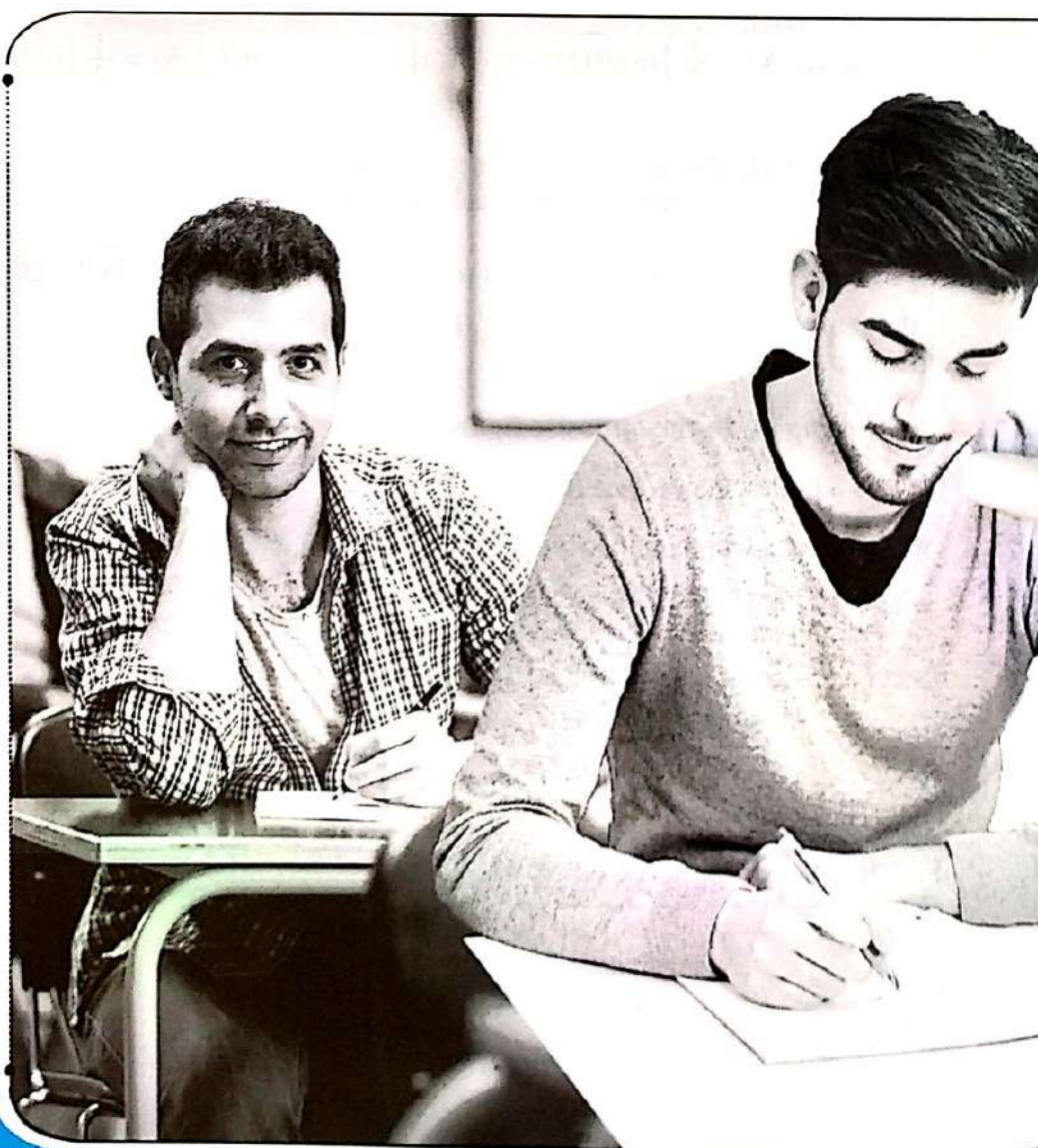
where $MA = 7$ cm. , then $P_M(A) = 7^2 - 5^2 = 24$



If $\begin{cases} \rightarrow P_M(A) > 0, \text{ then } \rightarrow A \text{ lies outside the circle M} \\ \rightarrow P_M(A) = 0, \text{ then } \rightarrow A \text{ lies on the circle M} \\ \rightarrow P_M(A) < 0, \text{ then } \rightarrow A \text{ lies inside the circle M} \end{cases}$

If A lies outside the circle M , then :	If A lies inside the circle M , then :
 $P_M(A) = AB \times AC = \vec{AB} \times \vec{AC} = (\vec{AD})^2$	 $P_M(A) = -AB \times AC = -\vec{AB} \times \vec{AC}$

Final Examinations of Geometry



School book examinations

Model 1

1 Complete the following :

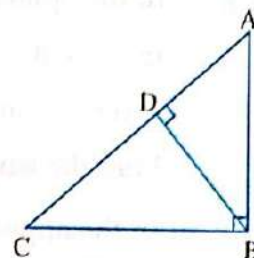
(1) The two polygons that are similar to a third are

(2) In the opposite figure :

First : $(AB)^2 = AD \times \dots\dots\dots$ and $(CB)^2 = CA \times \dots\dots\dots$

Second : $DA \times DC = \dots\dots\dots$

Third : $AB \times BC = \dots\dots\dots \times \dots\dots\dots$



2 Choose the correct answer from the given ones :

(1) Two similar rectangles , the length of the first is 5 cm. and the length of the second is 10 cm., then the ratio between the perimeter of the first to the perimeter of the second equals

(a) 1 : 5

(b) 1 : 3

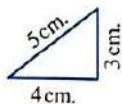
(c) 1 : 2

(d) 2 : 1

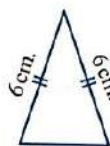
(2) Which two triangles of the following are similar ?



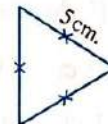
(1)



(2)



(3)



(4)

(a) (3) , (4)

(b) (1) , (3)

(c) (2) , (4)

(d) (1) , (4)

(3) If the ratio between the perimeters of two similar triangles is 1 : 4 , then the ratio between their two surface areas equals

(a) 1 : 2

(b) 1 : 4

(c) 1 : 8

(d) 1 : 16

(4) In the opposite figure :

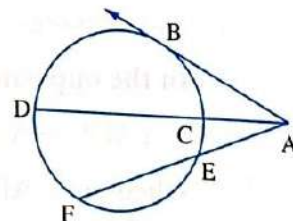
All the following mathematical expressions are correct except the expression

(a) $(AB)^2 = AC \times AD$

(b) $(AB)^2 = AE \times AF$

(c) $AC \times AD = AE \times AF$

(d) $AC \times CD = AE \times EF$

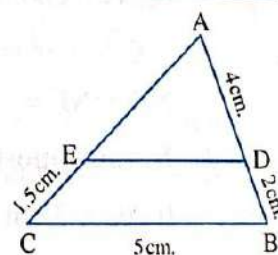


3 [a] In the opposite figure :

$\triangle ADE \sim \triangle ABC$ Prove that : $\overline{DE} \parallel \overline{BC}$

If $AD = 4$ cm. , $DB = 2$ cm. , $EC = 1.5$ cm.

, $BC = 5$ cm. , find the lengths of : \overline{AE} and \overline{DE}



[b] ABC is a triangle, $D \in \overline{BC}$ where $BD = 5$ cm.

, $DC = 3$ cm. and $E \in \overline{AC}$ where $AE = 2$ cm. , $CE = 4$ cm.

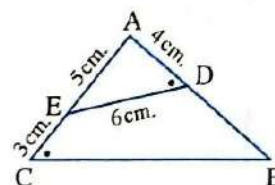
Prove that : $\triangle DEC \sim \triangle ABC$, then find the ratio between their two surface areas.

4 [a] In the opposite figure :

$$m(\angle ADE) = m(\angle C)$$

, $AD = 4$ cm. , $AE = 5$ cm. , $DE = 6$ cm. and $EC = 3$ cm.

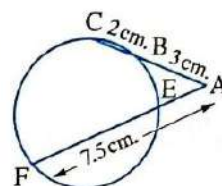
Find the lengths of : \overline{DB} and \overline{BC}



[b] In the opposite figure :

$\overline{CB} \cap \overline{FE} = \{A\}$, $AB = 3$ cm. , $BC = 2$ cm. , $AF = 7.5$ cm.

Find the length of : \overline{EF}



5 [a] \overline{AD} is a median in the triangle ABC , $\angle ADB$ is bisected by a bisector to cut \overline{AB} at E , $\angle ADC$ is bisected by a bisector to cut \overline{AC} at F and \overline{EF} is drawn.

Prove that : $\overline{EF} \parallel \overline{BC}$

[b] In the opposite figure :

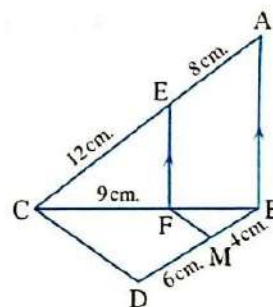
$\overline{AB} \parallel \overline{EF}$, $AE = 8$ cm.

, $CE = 12$ cm. , $CF = 9$ cm.

, $BM = 4$ cm. and $DM = 6$ cm.

(1) Find the length of : \overline{BF}

(2) Prove that : $\overline{FM} \parallel \overline{CD}$



Model (2)

1 Complete the following :

(1) Any two regular polygons that have the same number of sides are

(2) In the opposite figure :

If $\triangle ADE \sim \triangle ACB$

, then $m(\angle ADE) = m(\angle \dots\dots\dots)$

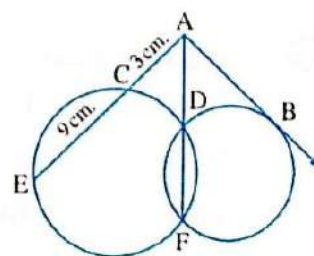
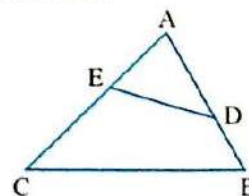
(3) If the two straight lines including the two chords \overline{DE}

, \overline{XY} intersect at the point N , then

$ND \times NE = \dots\dots\dots$

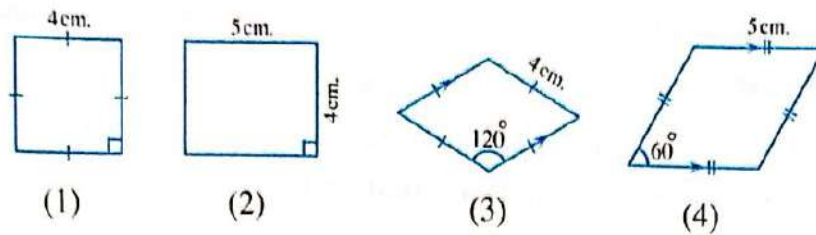
(4) In the opposite figure :

If $AC = 3$ cm. and $CE = 9$ cm. , then $AB = \dots\dots\dots$



2 Choose the correct answer from the given ones :

(1) Which two polygons of the following are similar ?



- (a) Polygons (1) , (2) (b) Polygons (1) , (3)
(c) Polygons (3) , (4) (d) Polygons (2) , (4)

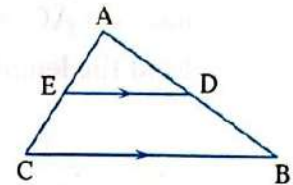
(2) If the ratio between the surface areas of two similar polygons is 16 : 25 , then the ratio between the lengths of two corresponding sides in the two polygons equals

- (a) 2 : 5 (b) 4 : 5 (c) 16 : 25 (d) 16 : 41

(3) In the opposite figure :

All the following mathematical expressions are correct except

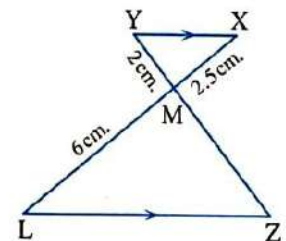
- (a) $\frac{AD}{DB} = \frac{AE}{EC}$ (b) $\frac{AD}{DB} = \frac{DE}{BC}$
(c) $\frac{AD}{AB} = \frac{AE}{AC}$ (d) $\frac{AB}{BD} = \frac{AC}{EC}$



(4) In the opposite figure :

The length of \overline{MZ} equals

- (a) 3.6 cm. (b) 4 cm.
(c) 4.2 cm. (d) 4.8 cm.

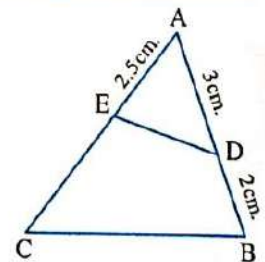


3 [a] In the opposite figure :

$$\triangle ABC \sim \triangle AED$$

Prove that :

BCED is a cyclic quadrilateral. If $AD = 3$ cm. , $BD = 2$ cm. and $AE = 2.5$ cm. , find the length of : \overline{EC}



[b] ABCD is a cyclic quadrilateral whose two diagonals intersected at E , \overline{EF} is drawn parallel to \overline{CB} to intersect \overline{AB} at F , \overline{EM} is drawn parallel to \overline{CD} to intersect \overline{AD} at M

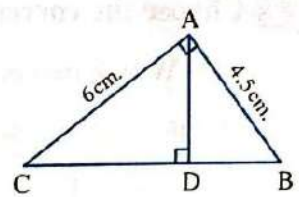
Prove that : $\overline{FM} \parallel \overline{BD}$

4 [a] In the opposite figure :

$$m(\angle BAC) = 90^\circ, \overline{AD} \perp \overline{BC}$$

, $AB = 4.5$ cm. and $AC = 6$ cm.

Find the length of each of : \overline{BD} , \overline{DC} and \overline{AD}

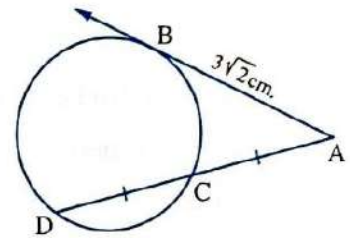


[b] ABCD is a cyclic quadrilateral in which : $BC = 27$ cm. , $AB = 12$ cm. , $AD = 8$ cm. , $DC = 12$ cm. and $AC = 18$ cm. **Prove that :** $\triangle BAC \sim \triangle ADC$ and find the ratio between their two surface areas.

5 [a] In the opposite figure :

\overline{AB} is a tangent to a circle , C is the midpoint of \overline{AD} and $AB = 3\sqrt{2}$ cm.

Find the length of : \overline{AC}



[b] ABC is a triangle in which : $AB = 8$ cm. , $AC = 12$ cm.

, $BC = 15$ cm. , \overline{AD} bisects $\angle A$ and intersects \overline{BC} at D , $\overline{DE} \parallel \overline{BA}$ is drawn to intersect \overline{AC} at E

Find the length of each of : \overline{BD} and \overline{CE}

Some schools examinations

1

Cairo Governorate

Nagr City Ed. Directorate
Al-Ola Language Modern Schools

Answer the following questions :

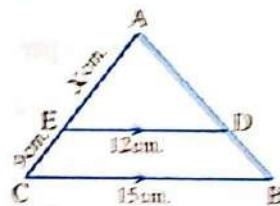
1 Choose the correct answer :

- (1) If the ratio of areas of two similar polygons is 4 : 9 , then the ratio of their perimeters
- (a) 9 : 4 (b) 4 : 9 (c) 2 : 3 (d) 3 : 2
- (2) If the point A where $AM = 8$ cm. and $r = 6$ cm. , then $P_M(A) =$
- (a) 10 (b) 18 (c) 40 (d) 28
- (3) The bisectors interior and the exterior of an angle of a triangle are
- (a) perpendicular. (b) parallel. (c) equal. (d) otherwise.

(4) In the opposite figure :

$x =$

- (a) 32 (b) 40
- (c) 36 (d) 10

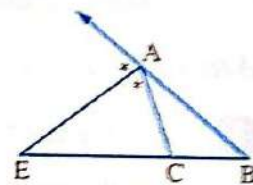


2 Complete :

- (1) Two polygons are similar if
- (2) The exterior bisector of the vertex angle of an isosceles triangle is to the base of the triangle
- (3) Regular polygons having the same number of angles are

(4) In the opposite figure :

\overline{AE} bisects $\angle BAC$ externally and intersects \overline{BC} at E.
 , then $AE =$



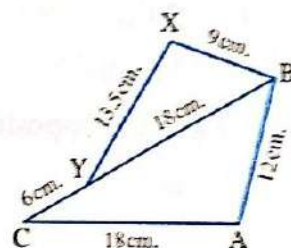
3 [a] If $\triangle ABC \sim \triangle XYZ$, and $3 AB = XY$, then find : $\frac{\text{area of } (\triangle ABC)}{\text{area of } (\triangle XYZ)}$

[b] In the opposite figure :

B , Y and C are collinear. $AB = 12$ cm. , $BX = 9$ cm.
 , $CY = 6$ cm. , $AC = BY = 18$ cm. , and $XY = 13.5$ cm.

Prove that : (1) $\triangle ABC \sim \triangle XBY$

(2) \overline{BC} bisects $\angle ABX$

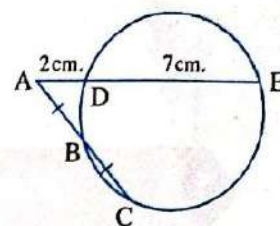


4 [a] In the opposite figure :

$AD = 2 \text{ cm.}$, $DE = 7 \text{ cm.}$

, $AB = BC$

, then find the length of : \overline{AC}



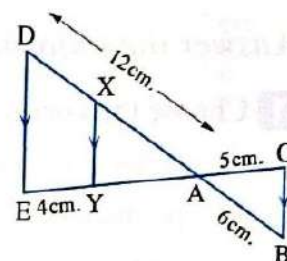
[b] In the figure opposite :

$\overline{XY} \parallel \overline{BC} \parallel \overline{DE}$

If $AB = 6 \text{ cm.}$, $AC = 5 \text{ cm.}$

, $AD = 12 \text{ cm.}$, $EY = 4 \text{ cm.}$

, then find the length of each of : \overline{AE} and \overline{DX}



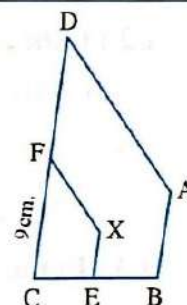
5 [a] In the opposite figure :

polygon $ABCD \sim$ polygon $XECF$

if $XE = \frac{1}{2} AB$

, $CF = 9 \text{ cm.}$

, then find the length of : \overline{FD}



[b] In the opposite figure :

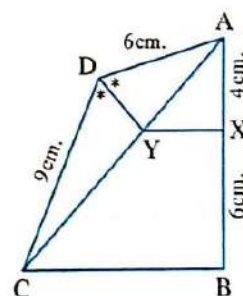
\overline{DY} bisects $\angle ADC$

, $DC = 9 \text{ cm.}$

, $DA = XB = 6 \text{ cm.}$

, $AX = 4 \text{ cm.}$

, then prove that : $\overline{YX} \parallel \overline{CB}$



2

Cairo Governorate

Western Cairo Educational Zone
Mathematics Inspection



Answer the following questions :

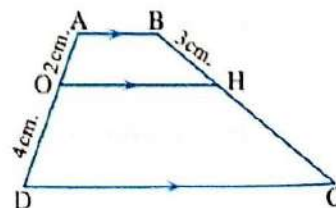
1 Complete the following :

(1) If the scale factor of similarity of two polygons = 1 , then the two polygon
are

(2) If $P_M(A) = 11$ and the length of the radius of the circle
M equal 5 cm. , then $AM = \dots\dots\dots \text{ cm.}$

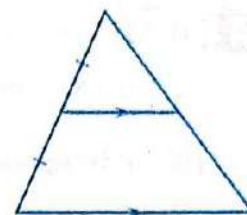
(3) In the opposite figure :

$BC = \dots\dots\dots \text{ cm.}$



(4) In the opposite figure :

In the surface area of the smaller triangle is 16 cm^2
 , then the surface area of
 the larger triangle = cm^2

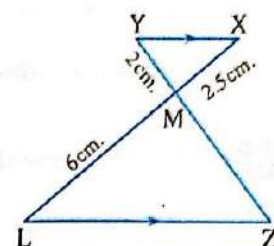


2 Choose the correct answer :

(1) Using the figure opposite :

Length of \overline{MZ} = cm.

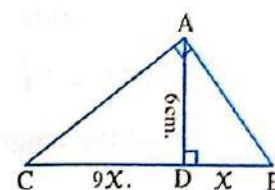
- (a) 3.6 (b) 4.2
 (c) 4 (d) 4.8



(2) In the opposite figure :

The value of X = cm.

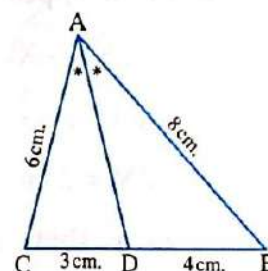
- (a) 2 (b) 4
 (c) 6 (d) 8



(3) In the opposite figure :

AD = cm.

- (a) 4 (b) 8
 (c) 6 (d) 5



**(4) If $\triangle ABC \sim \triangle DEF$, $m(\angle C) = 55^\circ$,
 $m(\angle B) = 80^\circ$, then $m(\angle D) =$ **

- (a) 55° (b) 80° (c) 45° (d) 40°

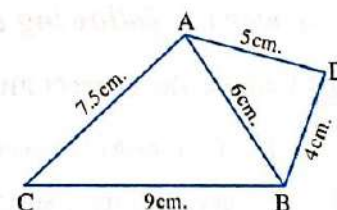
3 [a] In the opposite figure :

$AB = 6 \text{ cm}$, $BC = 9 \text{ cm}$, $AC = 7.5 \text{ cm}$.

$DB = 4 \text{ cm}$, and $DA = 5 \text{ cm}$,

Prove that : (1) $\triangle ABC \sim \triangle DBA$

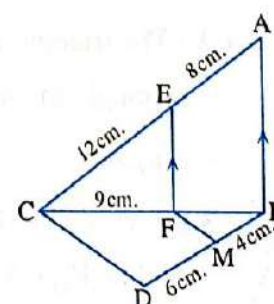
(2) \overrightarrow{BA} bisects $\angle DBC$



[b] In the opposite figure :

(1) Find the length of : \overline{BF}

(2) Prove that : $\overline{FM} \parallel \overline{CD}$



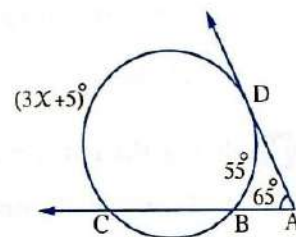
- 4 [a] \overline{AD} is a median in the triangle ABC , $\angle ADB$ is bisected by a bisector to cut \overline{AB} at E , $\angle ADC$ is bisected by a bisector to cut \overline{AC} at F and \overline{EF} is drawn. **Prove that : $\overline{EF} \parallel \overline{BC}$**

[b] In the opposite figure :

If $m(\angle A) = 65^\circ$, $m(\widehat{DB}) = 55^\circ$

, $m(\widehat{DC}) = (3x + 5)^\circ$

Find the value of : x



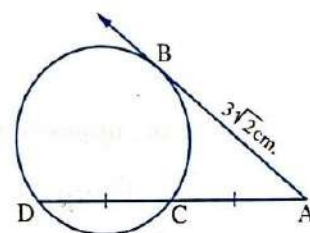
- 5 [a] In the opposite figure :

\overline{AB} is a tangent to a circle,

C is the midpoint of \overline{AD}

and $AB = 3\sqrt{2}$ cm.

Find the length of : \overline{AC}



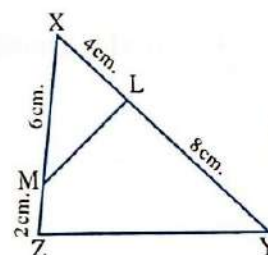
[b] In the opposite figure :

$L \in \overline{XY}$, $XL = 4$ cm., $YL = 8$ cm.

, $M \in \overline{XZ}$, $XM = 6$ cm.

, $ZM = 2$ cm.

Prove that : $LYZM$ is a cyclic quadrilateral.



3

Cairo Governorate

El Basateen and Dar Elsalam
Education Directorate



Answer the following questions :

- 1 Choose the correct answer :

(1) If the ratio between the perimeters of two similar polygons is $1 : 4$, then the ratio between their surface areas is

- (a) $1 : 2$ (b) $1 : 4$ (c) $1 : 8$ (d) $1 : 16$

(2) The triangle in which the measures of two angles are 35° and 75° is similar to the triangle in which the measures of two angles are 70°

- (a) 70° (b) 30° (c) 35° (d) 90°

(3) If M is a circle whose radius length is 6 cm., A is a point where $AM = 8$ cm., then $P_M(A) = \dots\dots\dots$

- (a) 10 (b) 18 (c) 40 (d) 28

(4) By using the opposite figure :

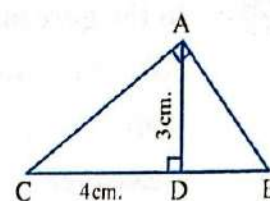
Area of $\triangle BAD$: Area of $\triangle BCA$ = :

(a) 3 : 4

(b) 4 : 5

(c) 3 : 5

(d) 9 : 25



2 Complete the following :

(1) The exterior and interior bisectors of an angle of a triangle are

(2) In the opposite figure :

$AB = 4$ cm. , $BC = 8$ cm.

and $AD = 6$ cm.

, then $DE =$ cm.

(3) In the opposite figure :

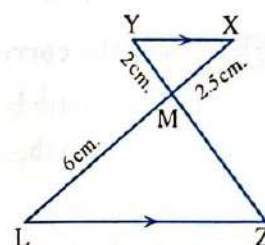
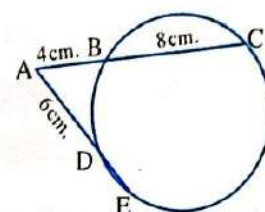
$XM = 2.5$ cm. , $YM = 2$ cm.

and $ML = 6$ cm.

, then $MZ =$ cm.

(4) If two straight lines intersect several parallel straight lines ,

then the length of the corresponding segments on the transversals are



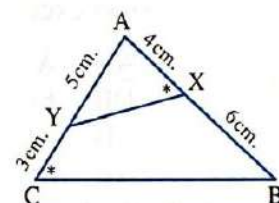
3 [a] In the opposite figure :

ABC is a triangle in which : $AX = 4$ cm.

, $XB = 6$ cm. , $AY = 5$ cm. , $YC = 3$ cm.

Prove that : (1) $\triangle AXY \sim \triangle ACB$

(2) $XBCY$ is a cyclic quadrilateral.

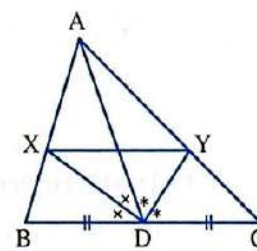


[b] In the opposite figure :

\overline{AD} is a median in $\triangle ABC$, $\overline{XY} \parallel \overline{BC}$

, \overline{DY} bisects $\angle ADC$ and intersect \overline{AC} at Y

prove that : \overline{DX} bisects $\angle ADB$



4 [a] ABC is triangle , $D \in \overline{AB}$ and $E \in \overline{AC}$ where :

$AD = 3$ cm. , $DB = 6$ cm. , $AE = 2$ cm. and $EC = 4$ cm.

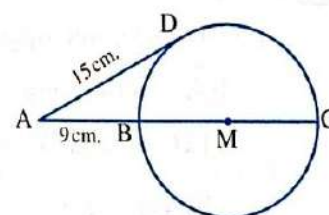
, prove that : $\overline{DE} \parallel \overline{BC}$

[b] In the opposite figure :

\overline{AD} is a tangent to the circle M at D

where $AD = 15$ cm. , and $AB = 9$ cm.

Calculate the radius length of the circle.



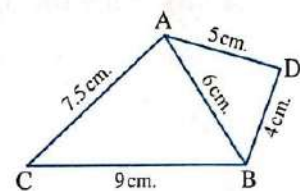
5 [a] In the figure opposite :

$AB = 6$ cm. $BC = 9$ cm. and $AC = 7.5$ cm.

, $DB = 4$ cm. and $AD = 5$ cm.

prove that : (1) $\triangle ABC \sim \triangle DBA$

(2) \overrightarrow{BA} bisects $\angle DBC$.



[b] If the power of point A with respect to the circle M = 144 where the radius length of the circle M is 5 cm. calculate the distance between the point A and the centre of the circle , then find the length of the tangent segment from the point A to the circle M.

4

Giza Governorate

Dokki District
Modern Narmar Language School



Answer the following questions :

1 Choose the correct answer :

(1) If the ratio between the perimeters of two similar triangles is $1 : 4$, then the ratio between their areas equals :

(a) $1 : 2$

(b) $1 : 4$

(c) $1 : 8$

(d) $1 : 16$

(2) In the figure opposite :

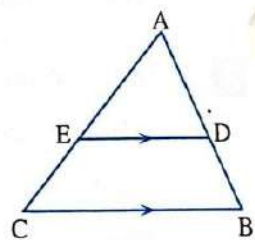
All the following expressions are correct except :

(a) $\frac{AD}{DB} = \frac{AE}{EC}$

(b) $\frac{AD}{DB} = \frac{DE}{BC}$

(c) $\frac{AD}{AB} = \frac{AE}{AC}$

(d) $\frac{AB}{BD} = \frac{AC}{EC}$



(3) In the figure opposite :

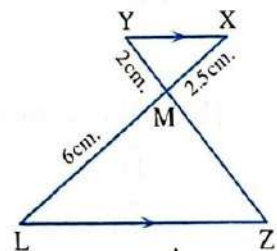
$MZ =$

(a) 3.6 cm.

(b) 4 cm.

(c) 4.2 cm.

(d) 4.8 cm.



(4) In the figure opposite :

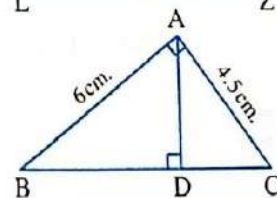
$CD =$

(a) 7.5 cm.

(b) 7.2 cm.

(c) 2.7 cm.

(d) 3.6 cm.



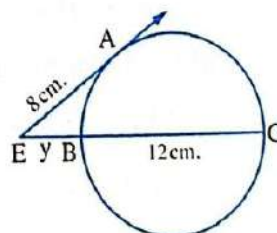
2 Complete :

(1) In the figure opposite :

\overrightarrow{EA} is a ray tangent to the circle. $EA = 8$ cm.

, $EB = y$ and $BC = 12$ cm.

, then $y =$ cm.

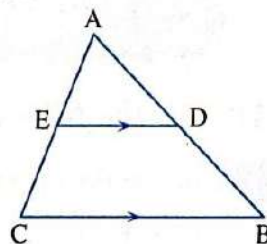


(2) If $\triangle ABC \sim \triangle XYZ$ and $AB = 3 XY$, then $\frac{\text{area}(\triangle XYZ)}{\text{area}(\triangle ABC)} = \dots\dots\dots$

(3) In the figure opposite :

$$\text{If } \frac{AE}{AC} = \frac{4}{7}$$

$$\text{, then } \frac{BD}{BA} = \dots\dots\dots$$

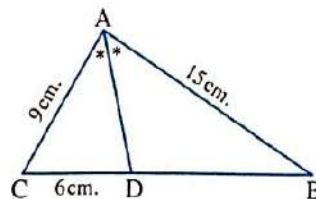


(4) In the figure opposite :

\overrightarrow{AD} is an angle bisector of $\angle BAC$, $AC = 9$ cm.

, $AB = 15$ cm. , $CD = 6$ cm. , $DB = x$ cm.

, then $x = \dots\dots\dots$ cm.



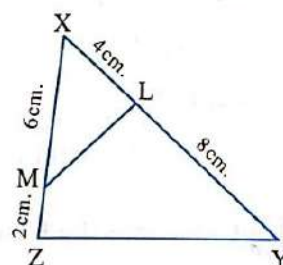
3 [a] In the figure opposite :

$\triangle XYZ$, $L \in \overline{XY}$ and $M \in \overline{XZ}$

where $XM = 6$ cm. , $ZM = 2$ cm.

prove that : (1) $\triangle XLM \sim \triangle XZY$

(2) $LYZX$ is a cyclic quadrilateral.

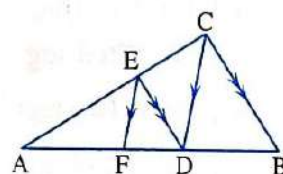


[b] In the figure opposite :

$\triangle ABC$ right angled at C.

, $\overline{BC} \parallel \overline{DE}$ and $\overline{CD} \parallel \overline{EF}$

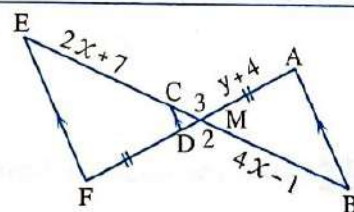
prove that : $AF \times AB = (AE)^2 + (ED)^2$



4 [a] In the figure opposite :

$\overline{AB} \parallel \overline{CD} \parallel \overline{EF}$

, then find the values of x and y . (all lengths are in cm).



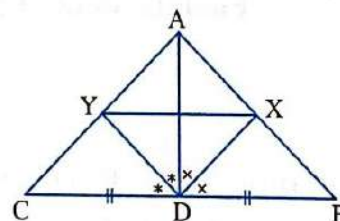
[b] In the figure opposite :

\overline{AD} is a median in $\triangle ABC$. \overline{DX} bisects $\angle ADB$,

intersects \overline{AB} at X. \overline{DY} bisects $\angle ADC$ and

intersects \overline{AC} at Y.

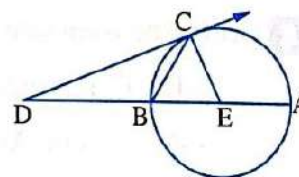
Prove that : $\overline{XY} \parallel \overline{BC}$.



5 [a] In the figure opposite :

\overline{DC} is tangent to circle E , $\frac{DB}{BE} = \frac{DC}{CE}$

Prove that : $\frac{DA}{DB} = \frac{AE}{BE}$ [Hint : Join \overline{AC}]



[b] ABCD is a quadrilateral , $E \in \overline{AC}$. \overline{EX} is drawn parallel to \overline{BC} intersects \overline{AB} at X.
 \overline{EY} is drawn parallel to \overline{CD} intersects \overline{AD} at Y. prove that : $AX \times AD = AB \times AY$

5

Giza Governorate

6th October directorate
Om El moamneen Language School



Answer the following questions :

1 Choose the correct answer :

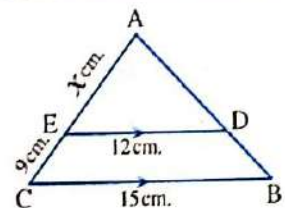
- (1) If $P_M(A) = 0$, then A lies the circle M.
 (a) on (b) inside (c) outside (d) otherwise
- (2) If $\Delta ABC \sim \Delta XYZ$ and $AB = 3XY$, then $\frac{\text{area of } (\Delta ABC)}{\text{area of } (\Delta XYZ)} = \dots\dots\dots$
 (a) 9 (b) 3 (c) $\frac{1}{3}$ (d) $\frac{1}{9}$
- (3) If the polygon $ABCD \sim \text{Polygon } XYZL$, then $AB \times ZL = XY \times \dots\dots\dots$
 (a) CD (b) BC (c) AB (d) AD
- (4) All are similar.
 (a) trapeziums (b) rhombuses (c) rectangles (d) squares

2 Complete the following :

- (1) If two straight lines intersects several parallel straight lines, then the lengths of the resulted segments are
- (2) Any two regular polygons that have the same number of sides are
- (3) Two polygons are similar if =,
- (4) The ratio between the lengths of two corresponding sides of two similar polygons is 2 : 3 if the perimeter of the smaller one is 14 cm., then the perimeter of the greater is cm.

3 [a] In the opposite figure :

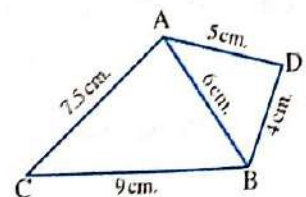
Find the value of : X



- [b] Polygon $ABCD \sim \text{polygon } XYZL$, if $AB = 32 \text{ cm.}$, $BC = 40 \text{ cm.}$, $XY = (3m - 1) \text{ cm.}$, $YZ = (3m + 1) \text{ cm.}$, find the numerical value of m

4 [a] In the opposite figure :

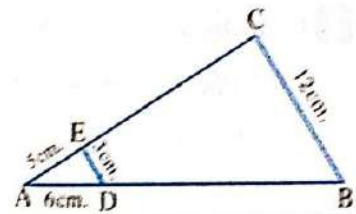
ABC is a triangle in which $AB = 6 \text{ cm.}$
 $BC = 9 \text{ cm.}$ $AC = 7.5 \text{ cm.}$, D is a point outside
 the triangle ABC. Where $DB = 4 \text{ cm.}$, $DA = 5 \text{ cm.}$
 Prove that : $\Delta ABC \sim \Delta DBA$



[b] In the opposite figure :

$$\triangle ADE \sim \triangle ABC$$

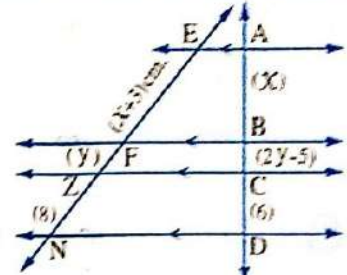
, find the length of : \overline{BD}



5 [a] In the opposite figure :

If $\overrightarrow{AE} \parallel \overrightarrow{BF} \parallel \overrightarrow{CZ} \parallel \overrightarrow{DN}$

Find the numerical value of each of : X , Y



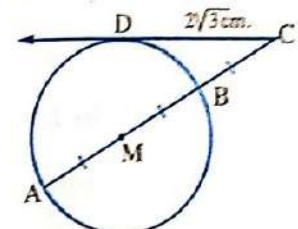
[b] In the opposite figure :

\overline{CD} is tangent to circle M

, $AM = MB = BC$.

, $DC = 2\sqrt{3}$ cm.

Find the diameter length of the circle M



6

Alexandria Governorate

East Educational Zone
Mathematics Directed



Answer the following questions : (Calculator is allowed)

1 Choose the correct answer :

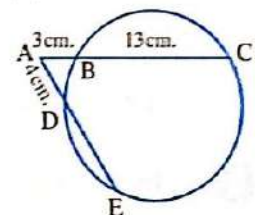
(1) The ratio between the two perimeters of two similar triangles is 2 : 3 , then the ratio between their areas is

- (a) 2 : 3 (b) 4 : 6 (c) 4 : 9 (d) 4 : 3

(2) By using the opposite figure :

The length of \overline{DE} equals

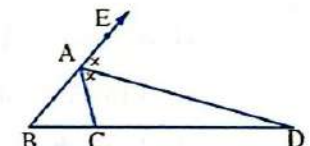
- (a) 6 (b) 8
(c) 10 (d) 12



(3) In the opposite figure :

$$\frac{BA}{AC} = \dots\dots\dots$$

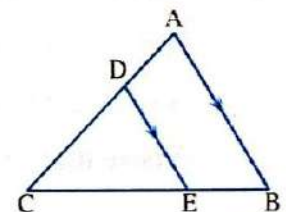
- (a) $\frac{AE}{AB}$ (b) $\frac{BD}{DC}$
(c) $\frac{AE}{AD}$ (d) $\frac{BC}{CD}$



(4) In the opposite figure :

$$\frac{CA}{AD} = \dots\dots\dots$$

- (a) $\frac{CE}{CD}$ (b) $\frac{AC}{CE}$
(c) $\frac{CB}{BE}$ (d) 1



2 Complete :

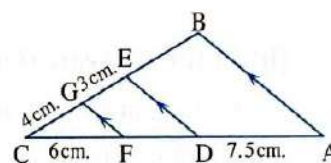
- (1) The interior and the exterior bisectors of an angle of a triangle are
- (2) If a line is drawn parallel to one side of a triangle and intersects the other two sides, then it
- (3) Any two regular polygons having the same number of sides are
- (4) In any right angled triangle, the altitude to the hypotenuse separates the triangle into

3 [a] In the opposite figure :

$\overline{AB} \parallel \overline{DE} \parallel \overline{FG}$, $CG = 4$ cm.

, $GE = 3$ cm. , $CF = 6$ cm. , $DA = 7.5$ cm.

, Find the length of : \overline{BE} , \overline{FD}

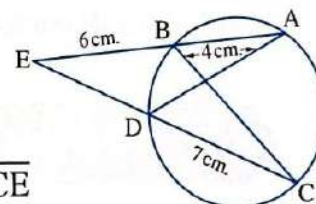


[b] In the opposite figure :

\overline{AB} and \overline{DC} are two chords in a circle , $\overline{AB} \cap \overline{CD} = \{E\}$,

$AB = 4$ cm. , $DC = 7$ cm. and $BE = 6$ cm.

, Prove that : $\triangle ADE \sim \triangle CBE$, then find the length of : \overline{CE}

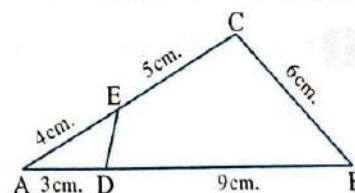


4 [a] In the opposite figure :

ABC is a triangle , $D \in \overline{AB}$, $E \in \overline{AC}$

(1) Prove that : $\triangle ADE \sim \triangle ACB$

(2) Find the length of : \overline{ED}



[b] In $\triangle ABC$, $AC > AB$, $M \in \overline{AC}$ where $m(\angle ABM) = m(\angle C)$

prove that : $(AB)^2 = AM \times AC$

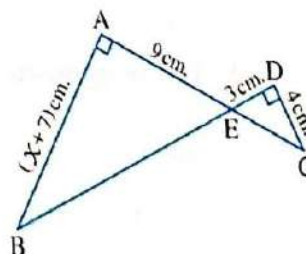
5 [a] In the opposite figure :

$\overline{BA} \perp \overline{AE}$, $\overline{CD} \perp \overline{DE}$, $AB = (X + 7)$ cm.

$AE = 9$ cm. , $ED = 3$ cm. , $DC = 4$ cm.

(1) Find the value of : X

(2) Find the length of : \overline{EB}

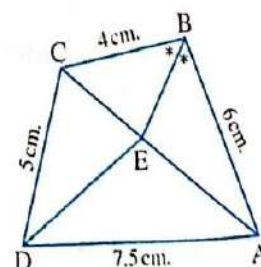


[b] In the opposite figure :

\overline{BE} bisects $\angle B$ and intersects \overline{AC} at E , $AB = 6$ cm.

, $CD = 5$ cm. , $DA = 7.5$ cm. and $BC = 4$ cm.

Prove that : \overline{DE} bisects $\angle ADC$.



7

Alexandria Governorate

Montazah Educational Zone
Frontiers Language School



Answer the following questions :

1 Choose the correct answer :

- (1) If $P_M(A)$ = zero , then A lies the circle M
(a) on (b) inside (c) outside (d) otherwise
- (2) If the ratio between the area of two similar polygons is 4 : 9 , then the ratio of their perimeters is
(a) 9 : 4 (b) 4 : 9 (c) 2 : 3 (d) 3 : 2
- (3) The interior and the exterior bisectors of an angle of a triangle are
(a) perpendicular. (b) parallel. (c) equal. (d) otherwise.

(4) In the opposite figure :

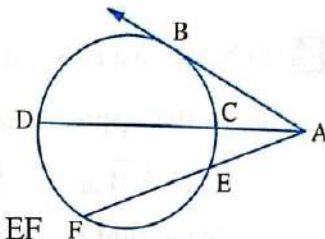
All mathematical expressions are correct except one expression

(a) $(AB)^2 = AC \times AD$

(b) $(AB)^2 = AE \times AF$

(c) $AC \times AD = AE \times AF$

(d) $AC \times CD = AE \times EF$



2 Complete each of the following:

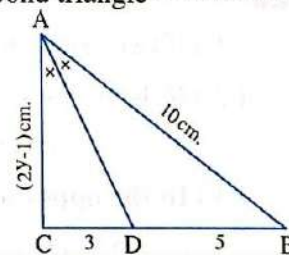
- (1) Any two regular polygons having the same number of sides are
- (2) In any right-angled triangle , the altitude to the hypotenuse separates the triangle into
- (3) The ratio between the lengths of two corresponding sides of two similar triangles is 2 : 5 , if the area of the first triangle = 24 cm^2 , then the area of the second triangle =

(4) In the opposite figure :

\overline{AD} bisects $\angle A$, $\frac{BD}{DC} = \frac{5}{3}$

, If $AB = 10 \text{ cm}$, $AC = (2y - 1) \text{ cm}$.

, then $y = \dots \text{ cm}$.

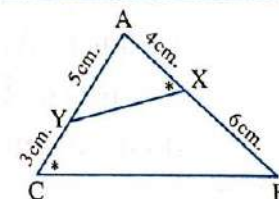


3 [a] In the opposite figure :

(1) prove that : $\triangle AXY \sim \triangle ACB$

(2) If the area of $\triangle AXY = 8 \text{ cm}^2$

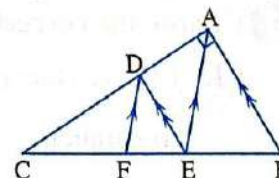
Find the area of the polygon XBCY



[b] In the opposite figure :

$\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AE}$

, prove that : $(CE)^2 = CF \times CB$



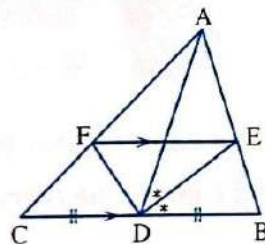
4 [a] In the opposite figure :

D is midpoint of \overline{BC} , \overline{DE} bisects $\angle ADB$, $\overline{EF} \parallel \overline{BC}$

Prove that : \overline{FD} bisects $\angle ADC$

If $DF = 4$ cm. , $DE = 5$ cm.

Find the length of : \overline{FE}



[b] In the opposite figure :

ABC is a triangle , $AB = 6$ cm.

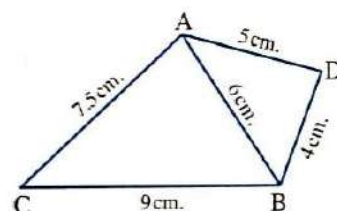
, $BC = 9$ cm. , $AC = 7.5$ cm.

, D is a point outside the triangle

where $BD = 4$ cm. , $AD = 5$ cm.

Prove that : (1) $\triangle ABC \sim \triangle DBA$

(2) \overline{BA} bisects $\angle DBC$



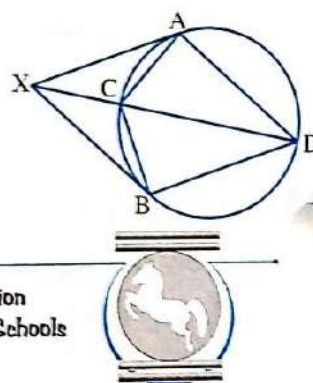
5 [a] State two cases of similarity of two triangles.

[b] In the opposite figure :

\overline{XA} , \overline{XB} are two tangent segments

Prove that : (1) $\triangle XBC \sim \triangle XDB$

(2) $BD \times AC = BC \times AD$



8

El-Sharkia Governorate

Directorate of Education
Dep. of Governmental L. Schools



Answer the following questions :

1 Complete the following :

(1) If two polygons are similar to a third one , then the two polygons are

(2) If the power of a point A with respect to the circle M is negative quantity , then A lies the circle.

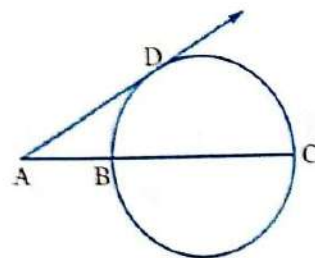
(3) In the opposite figure :

If \overline{AD} is a tangent and :

(a) If $m(\angle A) = 30^\circ$, $m(\widehat{BD}) = 45^\circ$

, then $m(\widehat{CD}) = \dots\dots\dots^\circ$

(b) If $AB = BC$, $AD = 3\sqrt{2}$ cm. , then $AC = \dots\dots\dots$



2 Choose the correct answer :

(1) The bisectors of angles of a triangle are

(a) parallel.

(b) concurrent.

(c) equal.

(d) perpendicular.

- (2) If the ratio between the perimeter of two similar polygons is 2 : 3 , then the ratio between their areas =

(a) 2 : 9 (b) 2 : 3 (c) 4 : 9 (d) 3 : 2

- (3) In the opposite figure if :

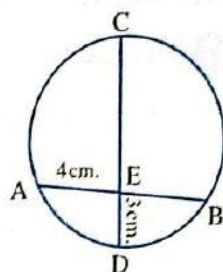
(i) $AE = 4 \text{ cm.}$, $AB = 10 \text{ cm.}$, $ED = 3 \text{ cm.}$
 , then $CD = \dots\dots\dots \text{ cm.}$

(a) 8 (b) 5

(c) 11 (d) 24

(ii) If $m(\angle AED) = 70^\circ$, $m(\widehat{AD}) = 50^\circ$, then $m(\widehat{BC}) = \dots\dots\dots^\circ$

(a) 70 (b) 90 (c) 100 (d) 140



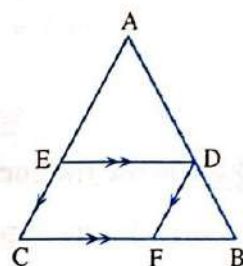
- 3 [a] In $\triangle ABC$ if $AB = 8 \text{ cm.}$, $AC = 6 \text{ cm.}$, $BC = 7 \text{ cm.}$, \overline{AD} bisect $\angle BAC$ and intersect \overline{BC} at D , find the length of : \overline{BD} and \overline{AD}

- [b] In the opposite figure :

$\overline{DE} \parallel \overline{BC}$

$\overline{DF} \parallel \overline{AC}$

Prove that : $\triangle ADE \sim \triangle DBF$



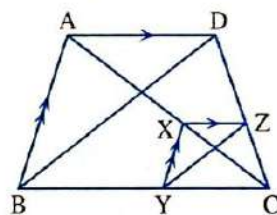
- 4 [a] ABC is a triangle , $D \in \overline{BC}$ where $(AC)^2 = CD \times CB$

Prove that : $\triangle ACD \sim \triangle BCA$

- [b] In the opposite figure if :

$\overline{XY} \parallel \overline{AB}$, $\overline{XZ} \parallel \overline{AD}$

Prove that : $\overline{YZ} \parallel \overline{BD}$



- 5 [a] In the opposite figure :

$\overline{AB} \cap \overline{CD} = \{E\}$, $AB = 5 \text{ cm.}$

, $CD = 9 \text{ cm.}$ and $ED = 3 \text{ cm.}$

Find the length of : \overline{BE}

- [b] In the opposite figure :

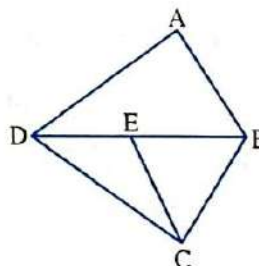
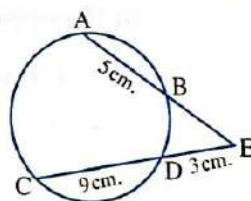
ABCD is a quadrilateral , $E \in \overline{BD}$ where :

$$\frac{AB}{DA} = \frac{CE}{BC} , \frac{BD}{DA} = \frac{EB}{BC}$$

Prove that :

(1) $\triangle ABD \sim \triangle CEB$

(2) $\overline{AB} \parallel \overline{CE}$



9

El-Gharbia Governorate

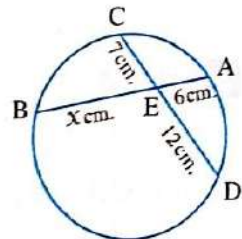
The Central Maths Supervision
Official Language Schools



Answer the following questions :

1 Complete :

- (1) In the isosceles Δ , the exterior bisector of the vertex angle of triangle is the base.
- (2) The ratio between the areas of two similar triangle is $16 : 25$, then the ratio between their perimeters is
- (3) If the point A where $AM = 8$ cm. and $r = 6$ cm. , then $P_M(A) = \dots\dots\dots$
- (4) In the opposite figure :
EA = 6 cm. CE = 7 cm. , ED = 12 cm. and BE = X cm.
; then X = cm.



2 Choose the correct answer :

(1) In the opposite figure :

\overline{AD} bisects exterior $\angle A$, then

(i) CD = cm.

(a) 2

(b) 6

(c) 4

(d) 8

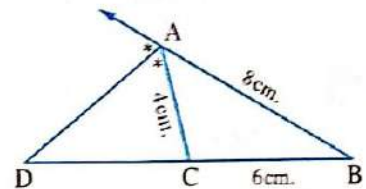
(ii) AD = cm.

(a) $2\sqrt{10}$

(b) 40

(c) $4\sqrt{10}$

(d) $10\sqrt{2}$



(2) In the opposite figure :

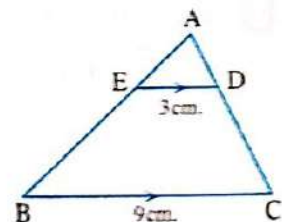
$\overline{DE} \parallel \overline{CB}$, AD : DC =

(a) 1 : 3

(b) 1 : 2

(c) 2 : 1

(d) 1 : 1



(3) If the area of $\Delta ABC = 45$ cm²

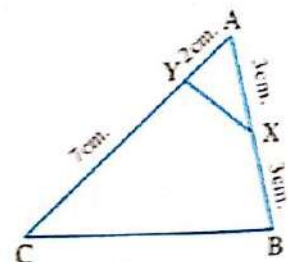
, then the area of : $\Delta AXY = \dots\dots\dots$ cm²

(a) 22.5

(b) 90

(c) 5

(d) 15

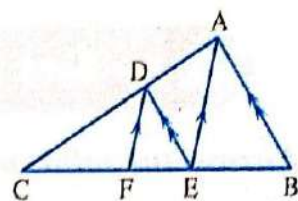


3 [a] In the opposite figure :

ABC is Δ , $D \in \overline{AC}$

, $\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AE}$

Prove that : $(CE)^2 = CF \times CB$



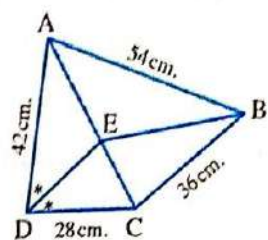
[b] In the opposite figure :

$AB = 54$ cm. $AD = 42$ cm.

, $DC = 28$ cm. and $BC = 36$ cm.

, \overline{DE} bisects $\angle ADC$.

Prove that : \overline{BE} bisects $\angle ABC$

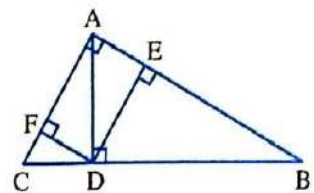


4 [a] In the opposite figure :

ABC is right-angled triangle at A , $\overline{AD} \perp \overline{BC}$

, $\overline{DE} \perp \overline{AB}$, $\overline{DF} \perp \overline{AC}$

Prove that : $\Delta ADE \sim \Delta CDF$

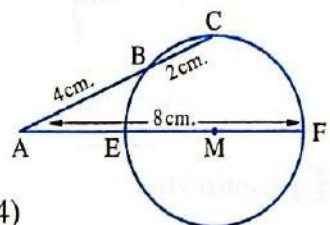


[b] In the opposite figure :

$\overline{CB} \cap \overline{FE} = \{A\}$, $AB = 4$ cm.

, $BC = 2$ cm. and $AF = 8$ cm.

Find the area and circumference of the circle where ($\pi = 3.14$)

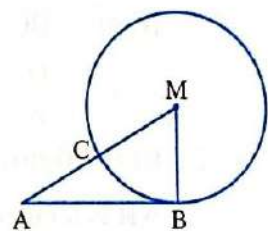


5 [a] In the opposite figure :

\overline{AB} is a tangent to the circle M at B . \overline{MA} intersects the circle M at C . If the radius length of the circle equals 12 cm. , $P_M(A) = 81$ cm. , then

Find : (1) The length of \overline{AB}

(2) The length of \overline{AC}



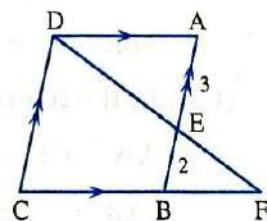
[b] In the opposite figure :

$ABCD$ is a parallelogram , $E \in \overline{AB}$

where $\frac{AE}{EB} = \frac{3}{2}$, $\overline{DE} \cap \overline{CB} = \{F\}$

(1) Prove that : $\Delta DCF \sim \Delta EAD$

(2) Find : $\frac{a(\Delta DCF)}{a(\Delta EAD)}$

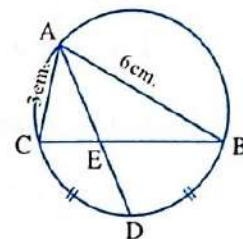




Answer the following questions :

1 Choose the correct answer :

- (1) If the ratio between the perimeters of two similar triangles is $1 : 4$, then the ratio between their two surface areas equals
- (a) $1 : 2$ (b) $1 : 8$ (c) $1 : 4$ (d) $1 : 16$
- (2) The measure of angle including between the two bisectors (interior and exterior) of an angle of a triangle equals
- (a) 60° (b) 90° (c) 30° (d) 45°
- (3) The power of point A with respect to circle M with radius length 4 cm. , $AM = 5$ cm. equals cm^2
- (a) 1 (b) 9
(c) 49 (d) zero
- (4) In the opposite figure :
- $AB = 6$ cm. , $AC = 3$ cm. , then $CE : CB =$
- (a) $1 : 2$ (b) $1 : 3$
(c) $3 : 1$ (d) $2 : 1$



2 Complete :

- (1) In the opposite figure :

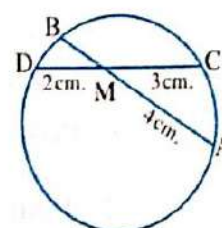
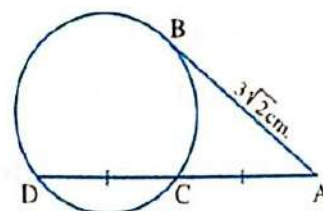
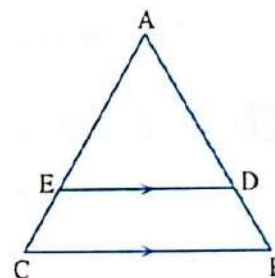
If $\overline{DE} \parallel \overline{BC}$
 , then $\frac{AD}{DB} = \frac{\dots\dots\dots}{\dots\dots\dots}$

- (2) In the figure opposite :

\overline{AB} is a tangent
 , C is a midpoint on \overline{AD}
 , $AB = 3\sqrt{2}$ cm.
 , then $AC =$

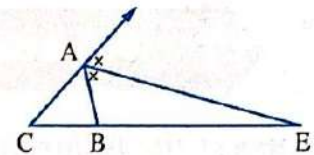
- (3) In the figure opposite :

$\overline{AB} \cap \overline{CD} = \{M\}$, $MA = 4$ cm.
 , $MC = 3$ cm. , $MD = 2$ cm.
 , then the length of $\overline{BM} =$ cm.



(4) In the opposite figure :

$$\frac{AC}{AB} = \frac{\dots\dots\dots}{\dots\dots\dots}$$

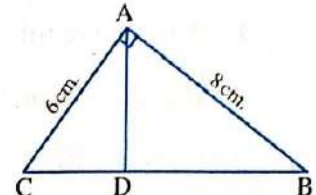


3 [a] In the opposite figure :

$$\triangle DBA \sim \triangle ABC, m(\angle BAC) = 90^\circ$$

(1) prove that : $\overline{AD} \perp \overline{BC}$

(2) If $AB = 8$ cm. , $AC = 6$ cm. , find the length of : \overline{BD}

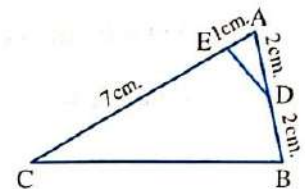


[b] In the opposite figure :

$$AD = BD = 2 \text{ cm. , } AE = 1 \text{ cm. , } EC = 7 \text{ cm.}$$

Find area $\triangle ADE$: area $\triangle ACB$

Prove that : DBCE is cyclic quadrilateral.



4 [a] ABCD is a cyclic quadrilateral , if $\overline{BA} \cap \overline{CD} = \{E\}$

Prove that : (1) $\triangle EAD \sim \triangle ECB$ (2) $EA \times EB = ED \times EC$

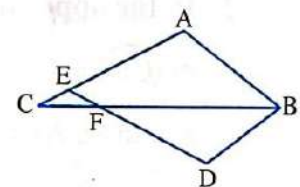
[b] In the opposite figure :

$$AB = 6 \text{ cm. , } BC = 12 \text{ cm. , } CA = 8 \text{ cm. , } FC = 3 \text{ cm.}$$

$$\text{, } DB = 4.5 \text{ cm and } DF = 6 \text{ cm.}$$

Prove that : (1) $\triangle ABC \sim \triangle DBF$

(2) $\triangle EFC$ is an isosceles triangle.

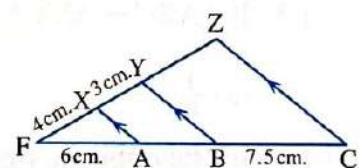


5 [a] In the opposite figure :

$$\overline{AX} \parallel \overline{BY} \parallel \overline{CZ}, XY = 3 \text{ cm.}$$

$$\text{, } FA = 6 \text{ cm. , } BC = 7.5 \text{ cm. , } FX = 4 \text{ cm.}$$

Find the length of each of : \overline{AB} , \overline{ZY}

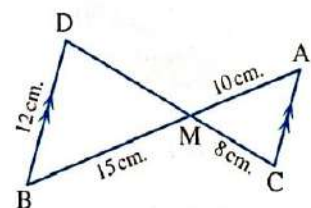


[b] In the opposite figure :

$$\overline{AC} \parallel \overline{DB}, AM = 10 \text{ cm. , } MB = 15 \text{ cm.}$$

$$CM = 8 \text{ cm. and } BD = 12 \text{ cm.}$$

Find the length of each : \overline{AC} , \overline{MD}



11

Damietta Governorate

Damietta Inspectorate of Mathematics
Official Language School

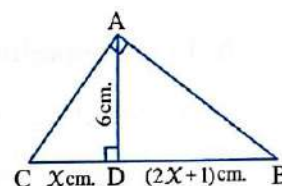


Answer the following questions :

1 Complete each of the following :

- (1) Any two regular polygons having the same number of sides are
- (2) The interior and the exterior bisectors of an angle of a triangle are
- (3) If $\overline{AC} \cap \overline{BD} = \{M\}$, and $MA \times MC = MB \times MD$, then the figure ABCD is
- (4) In the opposite figure :

$AD = 6$ cm. , $CD = X$ cm. , $BD = (2X + 1)$ cm.
 , then $X =$ cm.



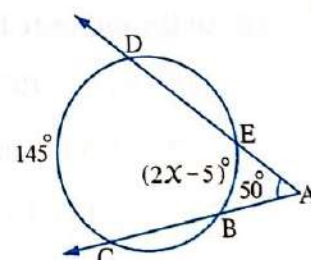
2 Choose the correct answer from those given :

- (1) If $P_M(A) = 0$, then : A lies the circle M
- (a) on (b) inside (c) outside (d) otherwise

(2) In the opposite figure :

$m(\widehat{CD}) = 145^\circ$, $m(\widehat{EB}) = (2X - 5)^\circ$
and $m(\angle A) = 50^\circ$, then $X =$ °

- (a) 80 (b) 50
(c) 25 (d) 15



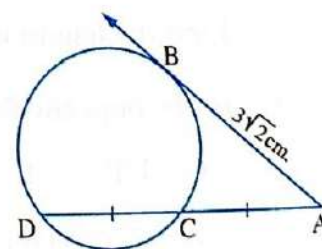
- (3) If $\triangle ABC \sim \triangle XYZ$ and $AB = 3 XY$, then $\frac{\text{area of } \triangle ABC}{\text{area of } \triangle XYZ} =$

- (a) $\frac{1}{9}$ (b) 9 (c) $\frac{1}{3}$ (d) 3

(4) In the opposite figure :

\overline{AB} is a tangent to the circle and C is midpoint of \overline{AD}
 , then $CD =$ cm.

- (a) 9 (b) 3
(c) $\frac{1}{3}$ (d) $\frac{1}{9}$



- 3 [a] ABC is a triangle in which $AB = 27$ cm. , $AC = 15$ cm. , \overline{AD} bisects $\angle A$ and intersects \overline{BC} at D where $BD = 18$ cm. , Calculate the length of each \overline{CD} and \overline{AD}

[b] In the opposite figure :

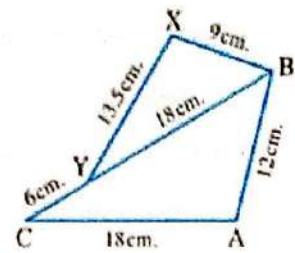
B, Y and C are collinear, $AB = 12$ cm, $BX = 9$ cm.

$CY = 6$ cm, $AC = BY = 18$ cm, and $XY = 13.5$ cm.

Prove that :

(1) $\triangle ABC \sim \triangle XBY$

(2) \overline{BC} bisects $\angle ABX$



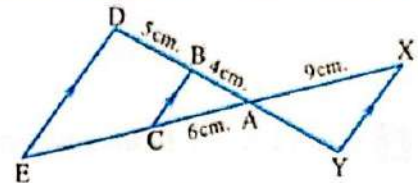
4 [a] In the opposite figure :

$\overline{XE} \cap \overline{YD} = \{A\}$, $B \in \overline{AD}$, $C \in \overline{AE}$

where : $\overline{XY} \parallel \overline{BC} \parallel \overline{DE}$

If $AC = 6$ cm, $AB = 4$ cm, $AX = 9$ cm, and $DB = 5$ cm.

Find the length of each of : \overline{AY} and \overline{EC}



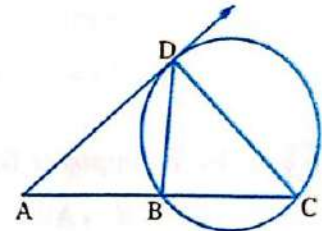
[b] In the opposite figure :

\overline{AD} is a tangent to the circle such that $\frac{DB}{DC} = \frac{1}{2}$

Prove that : $\triangle ADB \sim \triangle ACD$

and if the area of $\triangle ADB = 10$ cm²

, find the area of : $\triangle BDC$



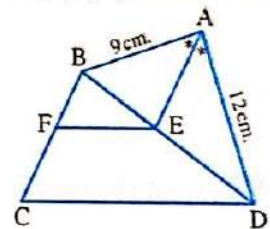
5 [a] In the opposite figure :

$AB = 9$ cm, $AD = 12$ cm.

\overline{AE} bisects $\angle BAD$, $F \in \overline{BC}$

such that : $4 BF = 3 FC$

Prove that : $\overline{FE} \parallel \overline{DC}$



[b] $\triangle ABC$, $D \in \overline{AB}$, $E \in \overline{AC}$ where : $AD = 3$ cm, $DB = 9$ cm, $AE = 4$ cm, and $CE = 5$ cm.

(1) Prove that : $\triangle AED \sim \triangle ABC$

(2) Prove that : $EDBC$ is a cyclic quadrilateral.

12

El-Beheira Governorate

Directory of Education
Mathematics Inspectorate



Answer the following questions :

1 Choose the correct answer from the given ones :

(1) The two polygons similar to a third are

(a) similar.

(b) congruent.

(c) rectangle.

(d) otherwise.

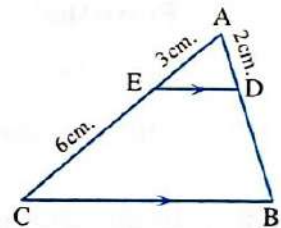
- (2) If A lies on the circle M, then $P_M(A)$ 0
 (a) < (b) \leq (c) > (d) =
- (3) The ratio between the perimeters of two similar polygons is $\tan^2 30^\circ : \cos 60^\circ$, then the ratio between their surface areas equals
 (a) 4 : 9 (b) 2 : 3 (c) 3 : 2 (d) 4 : 3
- (4) The bisectors of angles of a triangle are
 (a) perpendicular. (b) concurrent. (c) equal. (d) parallel.

2 Complete the following sentences :

- (1) If each one of two polygons is similar then ,
- (2) The interior and the exterior bisectors of an angle of a triangle at a vertex are
- (3) Two isosceles triangles are similar if
- (4) If $P_M(A) > 0$, then A lies

3 [a] In the opposite figure :

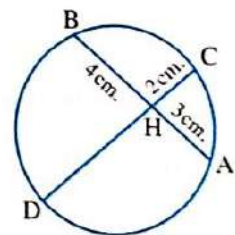
$\overline{DE} \parallel \overline{BC}$, $AE = 3$ cm.
 $EC = 6$ cm., $AD = 2$ cm.
 Find the length of : \overline{AB}



- [b] If $\triangle ABC \sim \triangle XYZ$ and the ratio between their perimeters is 3 : 4 and if the area of $\triangle XYZ$ is 32 cm^2 , then find the area of $\triangle ABC$

4 [a] In the opposite figure :

$\overline{AB} \cap \overline{DC} = \{H\}$, $CH = 2$ cm.
 $AH = 3$ cm. and $HB = 4$ cm.
 Find : DH

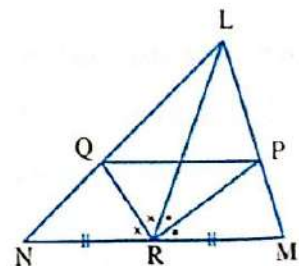


- [b] ABC is a triangle in which : $AB = 10$ cm. , $BC = 12$ cm. , $X \in \overline{AB}$
 where $AX = 4$ cm. , $Y \in \overline{BC}$ where $YC = 7$ cm. **Prove that :** $\triangle ABC \sim \triangle YBX$

- 5 [a] Determine the position of the point C with respect to the circle M if : $P_M(C) = -4$ and if the radius length of the circle M = 3 cm. , calculate CM**

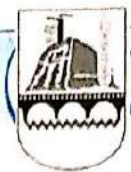
[b] In the opposite figure :

\overline{LR} is median , \overline{RP} bisects $\angle LRM$
 \overline{RQ} bisects $\angle LRN$
Prove that : $\overline{PQ} \parallel \overline{MN}$



13

Beni Suef Governorate

Directorate of Official Language
Education Administration

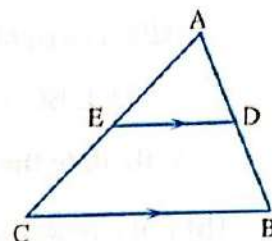
Answer the following questions : (Calculators are permitted)

1 Choose the correct answer :

(1) In the opposite figure :

If $\overline{ED} \parallel \overline{CB}$, $AD = 2$ cm. , $DB = 3$ cm.
and $AE = 4$ cm. , then $AC = \dots\dots\dots$ cm.

- (a) 3 (b) 4
(c) 6 (d) 10



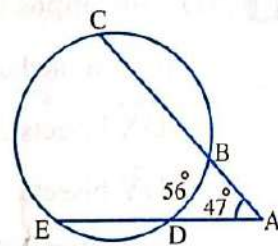
(2) The ratio between the lengths of two corresponding sides of two similar polygons is 3 : 4
if the perimeter of the smaller is 12 cm. , then the perimeter of the greater is $\dots\dots\dots$ cm.

- (a) 9 (b) 16 (c) 48 (d) 36

(3) In the opposite figure :

$m(\widehat{BD}) = 56^\circ$ and $m(\angle A) = 47^\circ$
 , then $m(\widehat{EC}) = \dots\dots\dots$

- (a) 90° (b) 140°
(c) 150° (d) 160°



(4) The measure of the angle lying between the interior and the exterior bisectors for any
angle of a triangle equals $\dots\dots\dots$

- (a) 45° (b) 90° (c) 135° (d) 180°

2 Complete :

(1) Any two regular polygons that have the same number of sides are $\dots\dots\dots$

(2) If the polygon $ABCD \sim$ the polygon $XYZL$, $\frac{AB}{XY} = \frac{1}{3}$

, then $\frac{\text{area of the polygon } ABCD}{\text{area of the polygon } XYZL} = \dots\dots\dots$

(3) Given several coplanar parallel lines and two transversals , then the lengths of the
corresponding segments on the transversals are $\dots\dots\dots$

(4) If the side lengths of two triangles are in proportion , then the two triangles are $\dots\dots\dots$

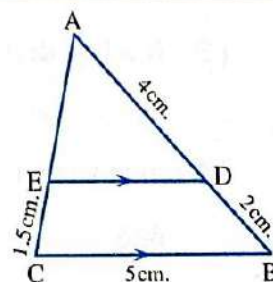
3 [a] In the opposite figure :

$\triangle ADE \sim \triangle ABC$, prove that : $\overline{DE} \parallel \overline{BC}$

If $AD = 4$ cm. , $DB = 2$ cm. , $EC = 1.5$ cm.

, $BC = 5$ cm.

Find the length of each of : \overline{AE} and \overline{DE}

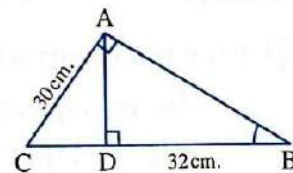


- [b] $\overline{XY} \cap \overline{ZL} = \{M\}$, where $\overline{XZ} \parallel \overline{LY}$, if $XM = 9$ cm. , $YM = 15$ cm.
and $ZL = 36$ cm. , Find the length of \overline{ZM}

4 [a] In the opposite figure :

ABC is a right-angled triangle at A
, $\overline{AD} \perp \overline{BC}$, $AC = 30$ cm. , $DB = 32$ cm.

Calculate the length of each of : \overline{CD} and \overline{AD}



- [b] If the power of a point A with respect to the circle M equals 144 where the radius length of the circle M equals 5 cm. , Calculate the distance between the point A and the center of the circle , then find the length of the tangent segment from the point A to the circle M

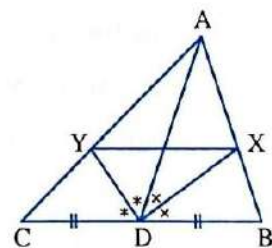
5 [a] In the opposite figure :

\overline{AD} is a median of $\triangle ABC$

, \overrightarrow{DX} bisects $\angle ADB$

, \overrightarrow{DY} bisects $\angle ADC$

Prove that : $\overline{XY} \parallel \overline{BC}$



- [b] Two circles are intersecting at A and B , $C \in \overline{AB}$ and $C \notin \overline{AB}$, from C the two tangent segments \overline{CX} and \overline{CY} are drawn to touch the circles at X and Y respectively.

Prove that : $CX = CY$

14

Assiut Governorate

L.S. Directorate
Math Inspection



Answer the following questions : (Calculator is allowed)

1 Complete the following :

- (1) Any two squares are
- (2) If $P_M(B) < 0$, then B lies
- (3) If a line drawn parallel to one side of triangle and intersects the other two sides , then it
- (4) In any right-angled triangle , the altitude to the hypotenuse divides the triangle into

2 Choose the correct answer :

(1) In the opposite figure :

$ED = 4 \text{ cm.}$, $BC = 12 \text{ cm.}$

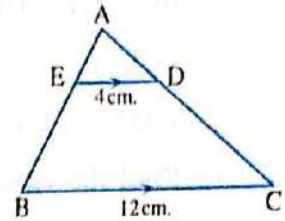
, then $\frac{AD}{AC} = \dots\dots\dots$

(a) 1 : 3

(b) 3 : 1

(c) 4 : 1

(d) 1 : 4



(2) If the ratio between perimeters of two similar polygons is 3 : 4 , then the ratio between their sides is

(a) 4 : 3

(b) 6 : 8

(c) 9 : 16

(d) 3 : 4

(3) In the opposite figure :

$AC = 9 \text{ cm.}$, $AB = 6 \text{ cm.}$, $BD = 4 \text{ cm.}$

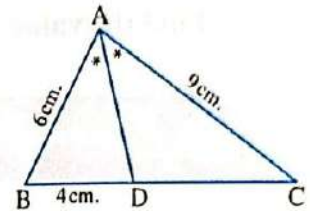
, then $BC = \dots\dots\dots \text{ cm.}$

(a) 12

(b) 16

(c) 8

(d) 10



(4) The measure of the angle between interior and exterior bisector of any angle of an equilateral triangle equals

(a) 135°

(b) 120°

(c) 90°

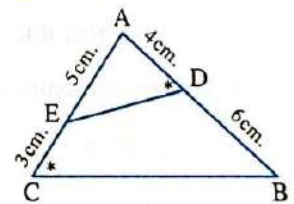
(d) 180°

3 [a] In the opposite figure :

$AD = 4 \text{ cm}$, $BD = 6 \text{ cm.}$

, $AE = 5 \text{ cm.}$, $EC = 3 \text{ cm.}$

Prove that : $\triangle ADE \sim \triangle ACB$



[b] \overline{AD} is a median in $\triangle ABC$, \overline{DE} bisects $(\angle ADB)$ and cuts \overline{AB} at E

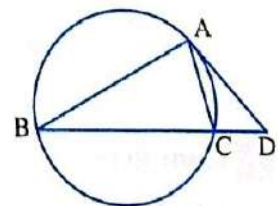
, \overline{DF} bisects $(\angle ADC)$ and cuts \overline{AC} at F. **Prove that : $\overline{EF} \parallel \overline{BC}$**

4 [a] In the opposite figure :

\overline{AD} is a tangent to the circle , $AB = 2 AC$

(1) **Prove that : $\triangle ACD \sim \triangle BAD$**

(2) If the area of $\triangle ACD = 12 \text{ cm}^2$, Find area of $\triangle BAD$



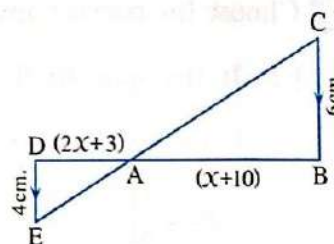
[b] In the opposite figure :

$\overline{BC} \parallel \overline{DE}$, $DE = 4$ cm.

, $CB = 6$ cm. , $AB = X + 10$

, $AD = 2X + 3$

Find the value of : X



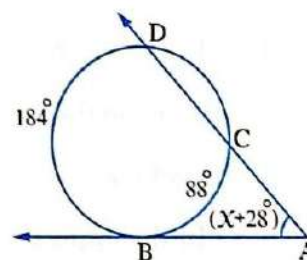
5 [a] In $\triangle ABC$, $D \in \overline{AB}$ where $AD = 2 BD$, $E \in \overline{AC}$ where $\overline{DE} \parallel \overline{BC}$, if the area of $\triangle ADE = 60 \text{ cm}^2$ Find the area of trapezium DBCE

[b] In the opposite figure :

If $m(\widehat{CB}) = 88^\circ$, $m(\widehat{BD}) = 184^\circ$

, $m(\angle A) = (X + 28)^\circ$

Find the value of : X



15

Aswan Governorate

Aswan Educational Directorate
Salam Private School



Answer the following questions :

1 Choose the correct answer :

(1) The ratio between the two perimeters of two similar triangles is 4 : 9 , then the ratio between their area is

(a) 4 : 9

(b) 2 : 3

(c) 16 : 81

(d) 9 : 4

(2) All the equilateral triangles are

(a) congruent.

(b) equal in perimeter.

(c) similar.

(d) equal in area.

(3) The measure of angle between the interior and exterior bisectors of any angle =

(a) 135°

(b) 90°

(c) 180°

(d) 45°

(4) In the opposite figure :

$AB = 12$ cm. , $CE = 4$ cm.

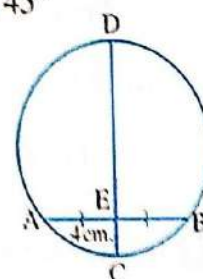
, then $ED =$

(a) 5

(b) 6

(c) 8

(d) 9

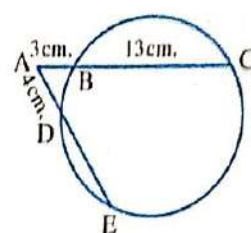


2 Complete :

(1) If the power of point A with respect to circle M is appositve quantity then the point A lies

(2) In the opposite figure :

DE =



(3) The exterior bisector of the vertex angle of an isosceles triangle to the triangle base.

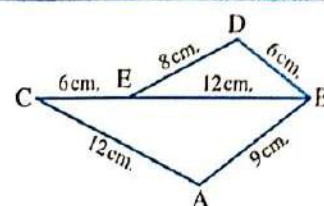
(4) If the scale factor of similarity of two polygons equals 1 then the two polygons are

3 [a] In the opposite figure :

B, E and C are collinear

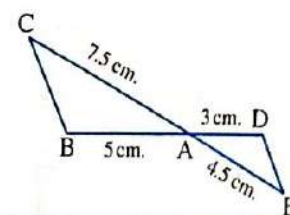
, prove that : (1) $\triangle ABC \sim \triangle DBE$

(2) \overrightarrow{BC} bisects $\angle ABD$



[b] In the opposite figure :

Prove that : $\overline{DE} \parallel \overline{BC}$



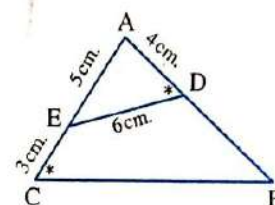
4 [a] In the opposite figure :

$m(\angle ADE) = m(\angle C)$

AD = 4 cm, AE = 5 cm, DE = 6 cm. and EC = 3 cm.

(1) Prove that : $\triangle ADE \sim \triangle ACB$

(2) Find the lengths of : \overline{DB} and \overline{BC}

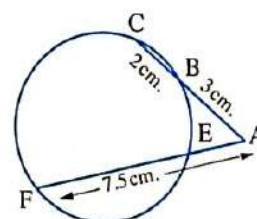


[b] In the opposite figure :

AB = 3 cm., BC = 2 cm.

, AF = 7.5

Find the length of : \overline{EF}



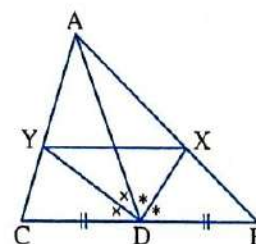
5 [a] In the opposite figure :

\overline{AD} is a median of $\triangle ABC$

\overrightarrow{DX} bisects $\angle ADB$

\overrightarrow{DY} bisects $\angle ADC$

Prove that : $\overline{XY} \parallel \overline{BC}$



[b] The ratio between the lengths of two corresponding sides in two similar triangles is 2 : 5, if the area of the smaller one is 16 cm^2 , find the area of the greater triangle.